

Conservation Laws and Thermodynamic Formalism for Dissipative Dynamical Systems

Inaugural Dissertation

zur Erlangung der philosophischen Doktorwürde

vorgelegt der

Philosophischen Fakultät II

der

Universität Zürich

von

Remo Badii

aus Firenze

Begutachtet von PD Prof. P.F. Meier und Prof. Dr. A. Thellung

Zürich 1987

Zentralstelle der Studentenschaft



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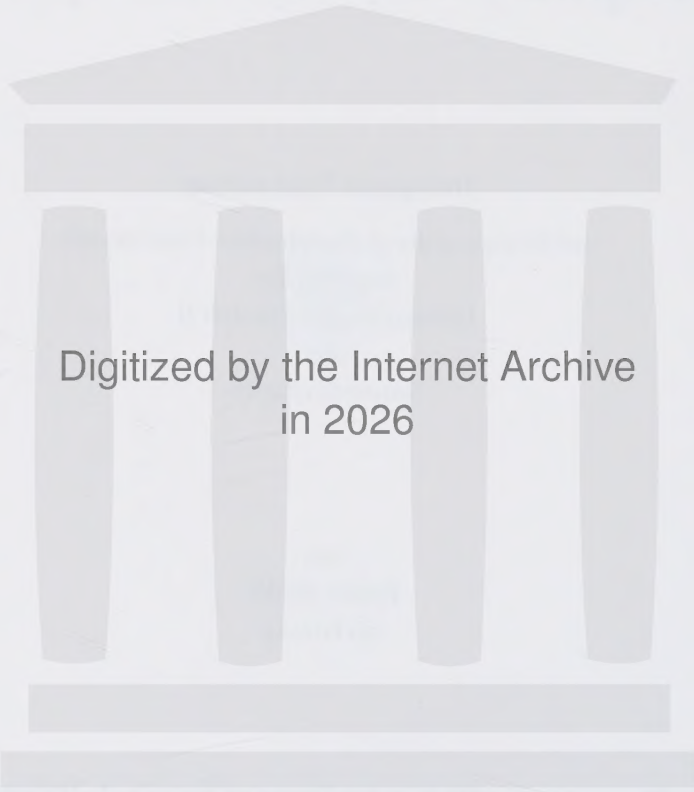
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to Gabi and my family

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Abstract

The theory of dissipative nonlinear dynamical systems is reviewed, in terms of invariant measures and symbolic dynamics. Generalized dimensions, metric entropies and Lyapunov exponents are defined and related to each other through suitable conservation laws. The latter can be expressed either by introducing pointwise quantities (local approach) or by evaluating global dynamical invariants like the dimension function $D(q)$ or the generalized metric entropies $K(q)$. The two points of view are formally equivalent to statistical mechanics and thermodynamics, respectively. The local approach constitutes the "microscopic" part of the theory, while the macroscopic description is obtained by specifying $D(q)$ and $K(q)$, which play the role of "thermodynamic" functions. Non-analytic behaviour of the dimension function $D(q)$ can be interpreted as a phase-transition of a one-dimensional spin model which is associated to the original nonlinear dynamical system through symbolic dynamics. The conditions under which such a behaviour can exist are analyzed. The lack of exact self-similarity is responsible for the appearance of edges in the "thermodynamic" function $D(q)$, since it produces long-range correlations in the spin chain. Fractal measures without self-similar structure can be generated in many ways: the most common source of such measures is represented by non-hyperbolic systems for which, however, no exact theory exists. Here, it is shown that long-range order in the time-lattice can be found in hyperbolic systems as well. To this end, a model of physical interest (a low-pass filtered hyperbolic map) which exhibits a "phase-transition" is introduced and solved analytically, by means of its symbolic-dynamical representation. The spin Hamiltonian is explicitly constructed. The lowest dimensionality of a hyperbolic map with non-analytic dimension function is three (two in the non-hyperbolic case).

Introduction

The interest in nonlinear systems has recently increased considerably, due to the richness of complex behaviour they can exhibit, even though a few simple deterministic dynamical equations are very often adequate to model them [1]. The time evolution, depending on the values of control parameters, can vary from an ordered motion (periodic or quasiperiodic solutions) to a very irregular behaviour. This phenomenon may already occur for systems with only three degrees of freedom and without external noise (*deterministic chaos*). In addition, it is not restricted to Hamiltonian problems but is common to dissipative systems as well. In the latter case, since volumes in phase-space are not conserved, but shrink to zero in time, the asymptotic solutions are called attractors.

The mechanisms which underly the onset of chaotic motion in many different fields of physics (fluid mechanics, lasers, chemical reactions, population dynamics, electronic circuits, etc.) are relatively well understood. Two generic routes to chaos have been discovered and thoroughly analyzed [2] (period-doubling [3] and quasiperiodicity transition [4]), in addition to interesting phenomena like intermittency [5]. The application of renormalization group ideas to these phenomena [6,7,8,9,10] allowed to identify universal quantitative features in the passage from ordered to (weakly) chaotic behaviour. A complete description of developed chaos is, however, much more difficult to achieve. Exact mathematical theories can only be formulated for special models which, often, do not faithfully represent physical systems, although providing good approximations of more generic cases. Statistical methods become, hence, particularly useful when the effect of nonlinearities is so important that no (mathematically) precise information about the solutions of the dynamical system are available. In particular, the evaluation of averages on ergodic invariant measures is a fundamental tool for the theoretical investigation in this field. The relevant quantities characterizing such measures are the generalized fractal dimensions [11]. They are related to information theory and refer to static properties (no time-dependence is considered): the fractal dimension roughly estimates the number of independent variables involved in the process. The dynamical properties of strange attractors are, instead, studied by means of metric entropy [12] and Lyapunov exponents [13,14]: these dynamical quantities measure the 'amount' of chaoticity of the attractor. The main property which makes them relevant in the description of strange attractors is their invariance under smooth coordinate changes.

Suitable conservation laws can be derived which connect dimensions and entropies with Lyapunov exponents, thus allowing a deeper understanding of the structure of the dynamical process. Such conservation laws can be formally rewritten in terms of partition functions for some hamiltonian spin-system and correspond to the adoption of different statistical mechanical ensembles (microcanonical, canonical, etc.).

The theory can be formulated both from a local approach (microscopic description) and from a global one (macroscopic): in the former case, pointwise

dimensions and entropies are considered as fluctuating functions of the position and their probability distributions are evaluated; in the latter case, the system is described by means of global indicators like the dimension function $D(q)$ and the generalized metric entropies $K(q)$. This is formally equivalent to statistical mechanics and thermodynamics, respectively. The exact correspondence between dynamical systems and one-dimensional statistical mechanics is obtained through symbolic dynamics, by interpreting time as the lattice coordinate. The relevance of this mapping comes from the occurrence of non-analytic behaviour in the "thermodynamic" function $D(q)$ for a large class of systems (non-hyperbolic systems). This can be interpreted as an evidence of a "phase-transition" on the time lattice. The absence of self-similarity is the necessary condition for the existence of such transitions and is related to long-time correlations in the symbol-sequences. In the present work, it is shown that non-hyperbolicity, although sufficient, is not necessary to produce fractal measures without exact self-similarity. Indeed, a three dimensional hyperbolic system exhibiting a phase-transition is introduced and analytically solved. The hamiltonian is constructed and the presence of long-range interactions between spins (symbols) evidenced. Such correlations, finally, arise from the tangency between eigendirections of the linearized map. Therefore, the lowest dimensionality in which it is possible to observe them is three, for hyperbolic maps, and two for non-hyperbolic ones.

In Section 1.1, dissipative dynamical systems are introduced; in Section 1.2, the invariant measures are discussed, while their characterization in terms of fractal dimensions is done in Section 1.3. Section 1.4 is devoted to metric entropies and Section 1.5 to Lyapunov exponents. The relations between dimensions, entropies and Lyapunov exponents are derived in Sections 1.6 and 1.7, where they are expressed as conservation laws for suitable generalized volumes. Finally, Section 1.8 deals with strange repellers, a kind of transient solutions that are of great interest for the understanding of chaotic systems.

In Part II, the mapping of dynamical systems onto one-dimensional spin models is developed. In Section 2.1, the microscopic description is presented. In Section 2.2, various statistical ensembles are constructed and related to different covering procedures of phase-space. In Section 2.3, the conditions for the occurrence of phase-transitions in one-dimensional lattices are reviewed and, in Section 2.4, an extension of the formalism to metric entropies, Lyapunov exponents and non-isotropic cases is performed.

In Part III, some systems exhibiting singularities in $D(q)$ or its derivatives are studied: in particular, a union of different Cantor sets is shown as a (non-physical) example of a zero-th order phase-transition (Section 3.1); the logistic and the Hénon maps are, then, studied as typical examples of non-hyperbolic transformations with non-self-similar invariant measure, in Section 3.2. Section 3.3 deals with the problem of filtering of chaotic signals and the change of the structure from self-similar to self-affine, when the filter's parameter is varied. In Section 3.4, finally, the filtered baker map (three-dimensional, hyperbolic) is explicitly solved and the associated phase-transition discussed.

Part I

Characterization of Chaotic Dynamical Systems

1.1 Dynamical Systems

Many physical systems can be modelled by evolution equations of the form

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \quad (1.1)$$

where $\mathbf{x}$ is a vector in the phase-space of dimension E , describing the state of the system. The nonlinear differentiable function $\mathbf{F}(\mathbf{x})$ determines the time behaviour of $\mathbf{x}(t)$. In case the time t is not continuous but assumes discrete values ($t = n$), a map is obtained:

$$\mathbf{x}_{n+1} = \mathbf{F}(\mathbf{x}_n). \quad (1.2)$$

We consider, in particular, dissipative dynamical systems, for which phase-space volumes are contracted by the time evolution (1.1, 1.2). This condition is verified when the divergence $\nabla \cdot \mathbf{F}$ of the flow (1.1) is negative, either everywhere or as a time-average along the trajectory. In the discrete-time case (1.2), dissipation occurs when the modulus of the Jacobian $|J| = \|\mathbf{DF}\|$ is less than 1. The symbol $\mathbf{DF}$ denotes the matrix of partial derivatives of $\mathbf{F}$. One-dimensional maps (i.e., F scalar) are dissipative if they are non-invertible since, in such a case, two or more points produce the same image point. In the following, discrete time will be considered, unless explicitly stated, since the study of maps is equivalent to the study of flows. In fact, e.g., a two-dimensional (invertible) map can be interpreted as a Poincaré map: i.e., as the intersection of the trajectory of a three-dimensional flow with some surface in phase-space. If, in turn, the solution of a two-dimensional map consists of points lying on a simple curve, the motion can be analyzed in terms of a one-dimensional recurrence for a suitable curvilinear coordinate x on that line. The map thus obtained will not be invertible, in general.

Due to the presence of dissipation, real experiments and computer simulations with dynamical systems exhibit transient behaviour followed by an asymptotic regime. In particular, given an initial open set U in phase-space, its n -th forward image $\mathbf{F}^n U$ will be arbitrarily close to a certain compact set A , for n large enough: that is, for every open set $V \supset A$, $\mathbf{F}^n U \subset V$, for $n \gg 1$. The set A is called *attracting set* if, in addition to the previous property, it is invariant under the application of the map: $\mathbf{F}^n A = A$, $\forall n$. The asymptotic solution $\mathbf{F}^n(\mathbf{x})$, however, will lie on a smaller set, called *attractor* (some parts of an attracting set, in fact, may not be attracting [1]). From an operational point of view, an attractor can be defined as the set on which points $\mathbf{x}_n = \mathbf{F}^n(\mathbf{x}_0)$ accumulate, for $n \rightarrow \infty$, under the effect of noise of amplitude σ , in the limit $\sigma \rightarrow 0$ (see [1], where also a more rigorous definition is given). The simplest examples of invariant sets which can be attractors are fixed points, periodic orbits (i.e., limit cycles, in the continuous-time case) and quasiperiodic orbits (tori). Such solutions are all regular (non-chaotic) since a small deviation $\delta\mathbf{x}_0$ on the initial condition remains small, for $n \rightarrow \infty$. When, instead, the uncertainty increases exponentially in time ($\delta\mathbf{x}_n = (\mathbf{DF}^n) \cdot \delta\mathbf{x}_0 \sim \delta\mathbf{x}_0 e^{n\lambda}$, $\lambda > 0$), the system displays sensitive dependence on initial conditions. This property can be chosen as a definition of *chaos*.

Attractors for which exponential separation occurs, along some directions, are called *strange*: notwithstanding the fact that volumes contract, the motion is unstable on the attractor. Since attractors are generally bounded, the exponential

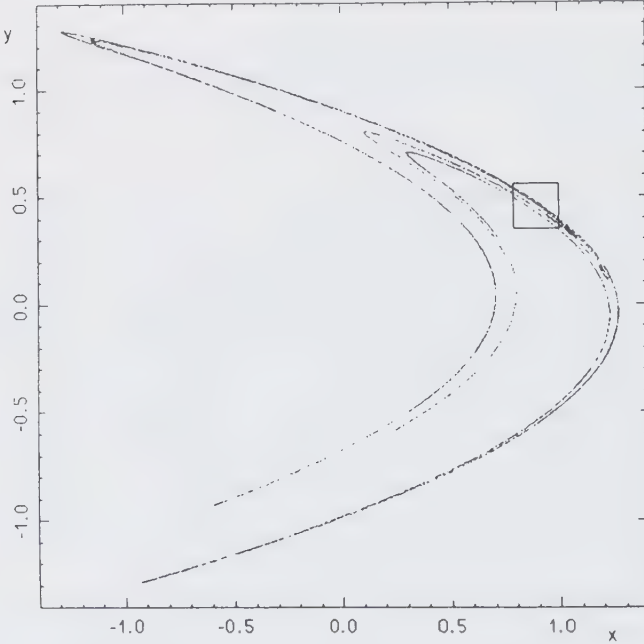


Figure 1: The Hénon attractor, for $a = 1.4$, $b = 0.3$.

divergency takes place at small distances only, and is followed by folding of the trajectory. As an example, consider the Hénon map [15] defined by

$$\begin{aligned} x_{n+1} &= 1 - ax_n^2 + by_n \\ y_{n+1} &= x_n, \end{aligned} \quad (1.3)$$

for $a = 1.4$ and $b = 0.3$. The corresponding strange attractor, shown in Fig. 1, has the aspect of a line folded onto itself many times, at the level of resolution of the figure, but its structure is actually richer, as it can be seen from the enlargement in Fig. 2, where a region in the neighbourhood of the fixed point $x^* = y^* = [b - 1 + \sqrt{(b - 1)^2 + 4a}]/(2a)$ is displayed. In fact, for small values of the parameter controlling the nonlinearity (a , in the case of the Hénon map), dynamical systems possess simple asymptotic solutions which become unstable when the parameter is gradually increased, giving rise to more complicated structures. As a consequence,

interspersed with a strange attractor, and arbitrarily close to it, there is a large number of repelling periodic orbits, remnants of the previously stable solutions. Strange attractors consist of (*global*) *unstable manifolds* of some unstable periodic orbit (the fixed point $\mathbf{x}^* \equiv (x^*, y^*)$, for the Hénon map). To define the invariant manifolds of system (1.2), consider the Jacobi matrix $D\mathbf{F}$ evaluated at $\mathbf{x}^*$, and its eigenvalues μ_1, μ_2 and eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. For the map (1.3), the modulus of the first eigenvalue μ_1 is larger than one and the corresponding eigendirection is unstable: i.e., points are repelled from $\mathbf{x}^*$ along that direction. Since areas are contracted at each iteration of the map by a factor $b = \|D\mathbf{F}\| = |\mu_1 \mu_2| < 1$, the second eigenvalue is smaller than one (in modulus). The *local stable and unstable*

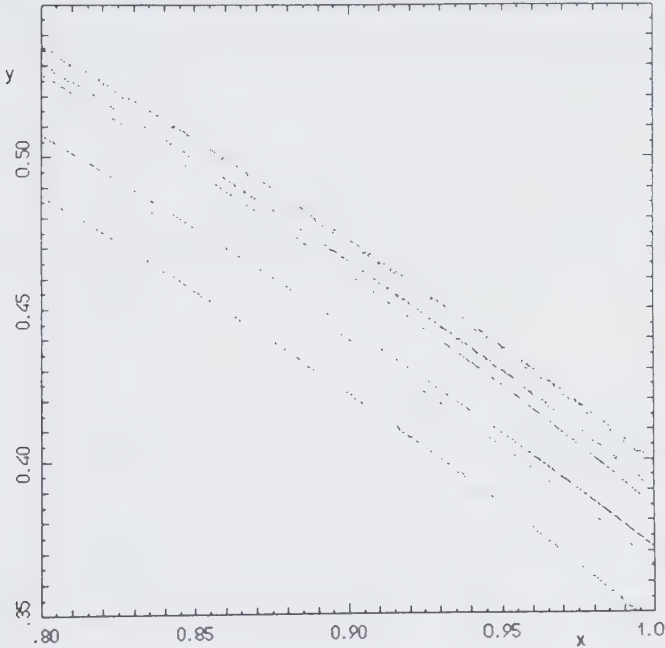


Figure 2: Enlargement of the region within the box in Fig. 1.

manifolds $W_{loc}^s(\mathbf{x}^*)$, $W_{loc}^u(\mathbf{x}^*)$ of $\mathbf{x}^*$ are, then, defined as [16]

$$W_{loc}^s(\mathbf{x}^*) = \{\mathbf{x} \in U \mid \mathbf{F}^n(\mathbf{x}) \rightarrow \mathbf{x}^* \text{ as } n \rightarrow +\infty, \text{ and } \mathbf{F}^n(\mathbf{x}) \in U, \forall n \geq 0\}, \quad (1.4)$$

$$W_{loc}^u(\mathbf{x}^*) = \{\mathbf{x} \in U \mid \mathbf{F}^n(\mathbf{x}) \rightarrow \mathbf{x}^* \text{ as } n \rightarrow -\infty, \text{ and } \mathbf{F}^n(\mathbf{x}) \in U, \forall n \leq 0\},$$

where U is a neighbourhood of the fixed point $\mathbf{x}^*$. The invariant manifolds W_{loc}^s and W_{loc}^u are the nonlinear extensions of the linear spaces E^s , E^u , spanned by

the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of the Jacobi matrix. At the point $\mathbf{x}^*$, W_{loc}^s and W_{loc}^u are tangent to E^s and E^u . The local manifolds W_{loc}^s and W_{loc}^u have global analogues W^s, W^u , obtained by iterating points in W_{loc}^s backwards in time and those in W_{loc}^u forwards:

$$\begin{aligned} W^s(\mathbf{x}^*) &= \bigcup_{n \leq 0} \mathbf{F}^n(W_{loc}^s(\mathbf{x}^*)), \\ W^u(\mathbf{x}^*) &= \bigcup_{n \geq 0} \mathbf{F}^n(W_{loc}^u(\mathbf{x}^*)). \end{aligned} \tag{1.5}$$

Similar definitions can be given for the continuous-time case (1.1). While the invariant manifolds of flows are composed of the unions of solution curves (trajectories are curves in phase-space), those of maps are unions of discrete orbit points. An attracting set A contains the unstable manifolds of all its points [1].

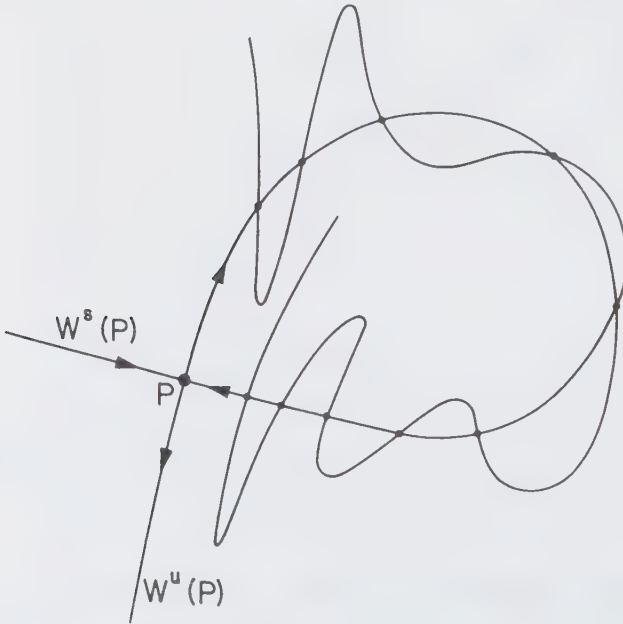


Figure 3: Schematic plot of the stable and unstable manifolds (W^s and W^u) of a saddle point (P). The homoclinic intersections are indicated by a small dot.

Existence and uniqueness of solutions of (1.1) forbid intersections between stable (or unstable) manifolds of different fixed points and self-intersections of $W^s(\mathbf{x}^*)$

or of $W^u(\mathbf{x}^*)$. The same holds for the discrete-time case if the map (1.2) is a diffeomorphism (i.e., if it has differentiable inverse). However, intersections of stable and unstable manifolds of distinct fixed points (*heteroclinic* intersection) or of the same fixed point (*homoclinic* intersection) can occur: they are a main source of the complex behaviour of dynamical systems. In Fig. 3, part of the stable and unstable manifolds of a saddle point P ($|\mu_1| > 1$ and $|\mu_2| < 1$) are shown. Since homoclinic points belong to invariant curves, they are mapped into other homoclinic points: the sequence thus obtained converges to P along the stable manifold. Due to the strong winding of $W^s(P)$ and $W^u(P)$ in the neighbourhood of P , a point in that region will be subjected to wildly erratic behaviour.

Strange attractors often exhibit a Cantor-set structure along directions transversal to the unstable manifolds, (see Fig. 2): i.e., they present the same aspect upon successive magnifications (*self-similarity* [11]). Such a property is related to the invariance of the set under the dynamical law (1.1) or (1.2). Fractal objects, however, can be generated by following various procedures and are not necessarily associated with chaotic behaviour. They can, therefore, be studied in terms of scaling properties of invariant measures, disregarding the underlying evolution law: this will be the subject of the next two Sections. The amount of “chaoticity” of the motion is instead given by dynamical quantities like metric entropies and Lyapunov exponents: relations between these and dimensions can be found (see Sections 1.6, 1.7, 1.8), thus yielding a more complete description of the phenomenology.

1.2 Invariant probability measures

The analysis of the long-time behaviour of dynamical systems is usually performed within the framework of *ergodic theory*: i.e., by evaluating suitable space averages which are equivalent to time averages. This condition is satisfied if the probability weight ρ , with which averages are taken, is an *invariant measure*:

$$\rho[\mathbf{F}^{-n}(U)] = \rho(U), \quad n > 0, \quad (1.6)$$

where U is a subset of phase-space and $\mathbf{F}^{-n}(U)$ is the n -th pre-image of U [1]. There are, in general, many invariant measures in a dynamical system, although not all physically relevant. For example, a δ -function centered on an unstable periodic

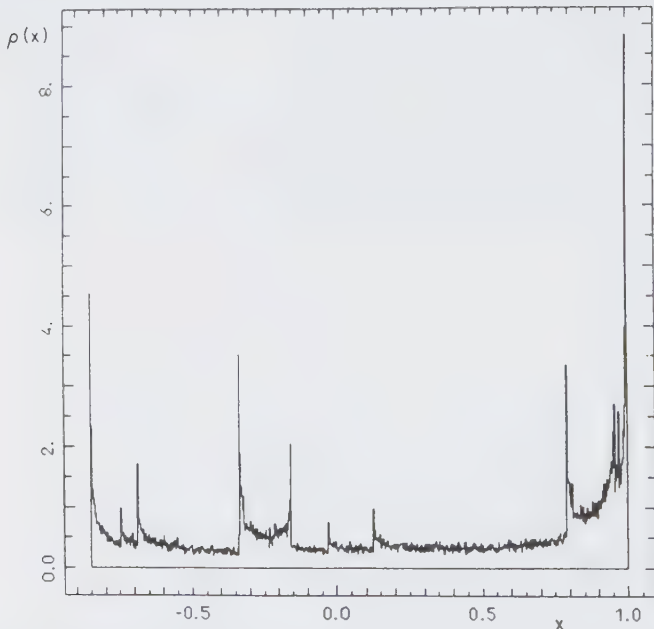


Figure 4: Invariant measure for the logistic map $x_{n+1} = 1 - 1.85x_n^2$, approximated by a histogram obtained by 10^5 iterates in 10^3 bins.

orbit is an invariant measure, but is not observed during the time evolution. From the point of view of experiments or computer simulations, a unique stationary measure ρ can be defined as the zero-noise limit of the measure ρ_σ obtained by the equations of motion (1.1) or (1.2), under the effect of a stochastic perturbation

of amplitude σ [1], in analogy with the operative definition of attractor (which is just the support of ρ). Such a measure is called *natural* [17] or *physical* [1] measure. Physically, ρ is assumed to represent experimental time averages. If the invariant probability measure cannot be decomposed into different pieces (each one still invariant), it is called *indecomposable* or *ergodic*. The physical measure

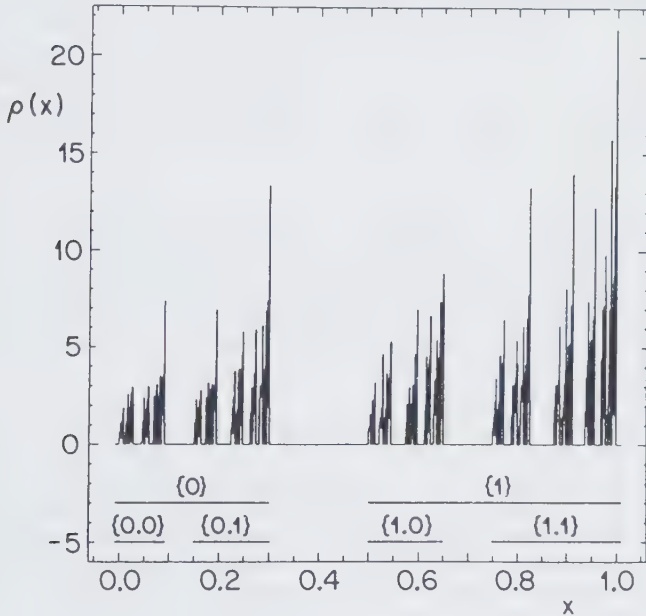


Figure 5: Histogram of 10^5 points in 10^3 bins for the non-uniform Cantor set (see Section 1.3), with $r_1 = 0.3$, $r_2 = 0.5$, $p_1 = 0.3$ and $p_2 = 0.7$. Some intervals are displayed with their symbol-sequence.

is assumed to be not only invariant, but also ergodic. In this case, the following holds (ergodic theorem):

$$\langle A \rangle \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T A[\mathbf{x}(t)] dt = \int A(\mathbf{x}) \rho(d^E \mathbf{x}), \quad (1.7)$$

where $A[\mathbf{x}(t)]$ is a continuous function and $d^E \mathbf{x}$ is the element of volume in phase-space, centered at $\mathbf{x}$. The position $\mathbf{x}(t)$, at time t , is the solution of Eq.(1.1) with initial condition $\mathbf{x}(0)$, chosen randomly with respect to ρ (in this way, zero probability is assigned to unstable periodic orbits). The measure $\rho(\mathbf{x})$ can be

expressed as a time average of delta functions:

$$\rho(\mathbf{x}) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \delta[\mathbf{x} - \mathbf{x}(t)] dt. \quad (1.8)$$

Other more refined definitions of measure can be given for dynamical systems. The main requirement they should satisfy, however, is smoothness along the unstable directions. Such measures are called SRB (from Sinai, Ruelle, Bowen) and usually coincide with physical measures, although (rather special) counterexamples exist [1]. While smoothness is guaranteed along the unstable manifolds by the stretching properties of chaotic systems, the measure can be highly singular in the transversal directions. In Fig. 4, the invariant measure of the logistic map

$$x_{n+1} = 1 - ax_n^2 \quad (1.9)$$

is displayed, for $a = 1.85$, as obtained from a histogram of 10^5 iterates over 10^3 bins. The probability density is smooth with square-root singularities at the forward images of the maximum ($x = 0$) and can essentially be considered to represent a typical measure along the unstable manifold of higher dimensional maps such as Hénon's map (1.3). For $a = 2$, the probability density $\rho(x)$ is analytically computable and is given by (see, for example, [18])

$$\rho(x) = \frac{1}{\pi \sqrt{1 - x^2}}. \quad (1.10)$$

An example of a highly singular distribution (like those along stable manifolds), is shown in Fig. 5: it corresponds to the non-uniform Cantor set to be defined in the next section. Some parts of the picture have a structure resembling that of the whole figure and contain, in turn, other sub-structures of the same form. Due to these self-similar properties, such distributions are called *fractal measures*. They will be the subject of the next section.

1.3 Dimensions

A self-similar object displays essentially the same features (geometric or probabilistic) at any level of magnification. The simplest example of a fractal object is the Cantor set C , composed of all points x which can be written in a ternary expansion containing only the digits 0 and 2 (not 1):

$$C = \{x \mid x = 2 \sum_{i=1}^{\infty} s_i 3^{-i}, s_i = 0, 1\}. \quad (1.11)$$

From a geometrical point of view, the set is obtained by deleting the middle third of the segment $[0, 1]$ and, then, by repeating the same procedure on each of the subsets thus generated. At the n -th step of this process, a number 2^n of intervals survive, each of length $\varepsilon_n = 3^{-n}$. Every interval (as well as any point x , for $n \rightarrow \infty$) is individuated by assigning its symbol sequence $\{s_i, i = 1, \dots, n\}$. The generation procedure for fractal sets does not necessarily derive from a dynamical system, but can be chosen arbitrarily. In fact, the digits 0 and 2 of the previous example could be selected with different probability weights, or the lengths of the intervals at each level of resolution n might not be all equal. A useful model for the study of strange attractors is the non-uniform Cantor set, constructed as follows: starting from the unit segment, the central part is removed and two intervals of lengths r_1 and r_2 are left at the extrema. The procedure is, then, iterated within each subinterval. In addition, different probability weights p_1, p_2 ($p_1 + p_2 = 1$), can be assigned to the two parts, at every step. In this way, at the n -th level of the process, there will be a number $\binom{n}{i}$ of segments of length $r_1^i r_2^{n-i}$ and weight $p_1^i p_2^{n-i}$, with $i \in [0, n]$ being the number of times the digit 0 occurs in the sequence of n bits which defines the interval (see Fig. 5). If the set is divided in M parts, instead of only three, the coefficients s_i will assume values in $[0, \dots, M-1]$.

Since Cantor sets on a line are composed of isolated points, their *topological* dimension D_T is zero: in fact, a set A is said to have topological dimension E if it is homeomorphic to R^E , i.e., if there is a one-to-one bi-continuous correspondence between its points and R^E (Cantor sets are homeomorphic to the set of natural numbers). A more useful definition of dimension is the following: the (compact) set A in an E -dimensional euclidean space is covered with boxes of size ε and the number $N(\varepsilon)$ of non-empty boxes is counted. The *box* or *fractal* dimension $D(0)$ is [11], then,

$$D(0) = -\lim_{\varepsilon \rightarrow 0} \frac{\ln \inf N(\varepsilon)}{\ln \varepsilon}, \quad (1.12)$$

where the presence of the infimum indicates that the most 'economic' covering should be taken (each box should contain the largest possible portion of the set, without overlapping with any other box). In computer calculations the infimum cannot be taken and a fixed regular grid is superimposed on the attractor. The two definitions, however, can be considered equivalent. In the case of the Cantor set C of Eq.(1.11), $N(\varepsilon_n) = 2^n$ and $\varepsilon_n = 3^{-n}$, yielding $D(0) = \ln 2 / \ln 3$. To be more precise, a generalized volume $V_\gamma(\varepsilon) = \varepsilon^\gamma \inf N(\varepsilon)$ should be introduced: it

will diverge, in the limit $\epsilon \rightarrow 0$, if $\gamma < D(0)$, while tending to zero, for $\gamma > D(0)$. Hence, the box dimension $D(0)$ can be defined (equivalently to Eq.(1.12)) as the infimum over all γ 's such that $V_\gamma(\epsilon) \rightarrow 0$, for $\epsilon \rightarrow 0$. A refinement of the concept of covering was proposed by Hausdorff [19], who considered balls of variable size ϵ_j , with the only constraint $\epsilon_j \leq \epsilon$. As a consequence, the volume $V_\gamma(\epsilon)$ is replaced by $\tilde{V}_\gamma(\epsilon) = \inf \sum_j \epsilon_j^\gamma$, where the infimum is taken over all coverings satisfying $\epsilon_j \leq \epsilon$. The Hausdorff dimension D_H is, in analogy with the box dimension, the infimum over all γ 's such that $\tilde{V}_\gamma(\epsilon)$ shrinks to zero, for $\epsilon \rightarrow 0$. An object is called *fractal* whenever its Hausdorff dimension D_H exceeds the topological dimension D_T [11]. Even though several examples exist in which $D(0) \neq D_H$, these two quantities usually coincide in all cases of interest in physics. Variants to the above definitions have been proposed, which differ only in very subtle aspects [11].

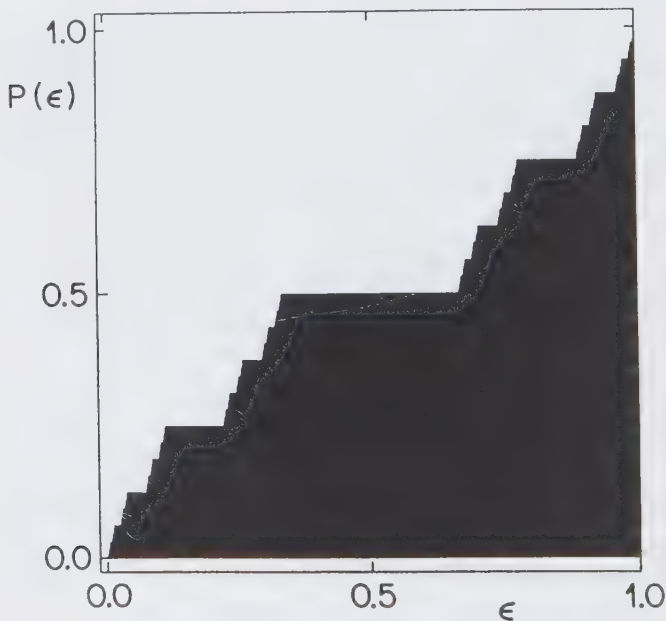


Figure 6: Mass integral $P(\epsilon)$ versus ϵ (Devil's Staircase) for the uniform Cantor set C (Eq.(1.11)).

The box dimension (1.12) only refers to geometrical properties and is not sensitive to the probabilistic structure present, for example, in the sets of Figs. 4 and 5. When dealing with fractals endowed with a measure ρ , the dimension is

introduced to describe the increase of mass with size. Indicating with

$$P(\varepsilon, \mathbf{x}) = \int_{B(\varepsilon, \mathbf{x})} d\rho(\mathbf{y}) \quad (1.13)$$

the mass contained in the ball (or box) $B(\varepsilon, \mathbf{x})$ of size ε centered at $\mathbf{x}$, the *pointwise* dimension [17] $\alpha(\mathbf{x})$ is defined as

$$\alpha(\mathbf{x}) = \lim_{\varepsilon \rightarrow 0} \frac{\ln P(\varepsilon, \mathbf{x})}{\ln \varepsilon}. \quad (1.14)$$

When $\alpha(\mathbf{x})$ actually depends on the position $\mathbf{x}$, the fractal is called *non-uniform*. It is useful to distinguish the pointwise dimension from the “crowding index” $\alpha(\mathbf{x}, \varepsilon)$ [30], defined in the same way, but without taking the limit $\varepsilon \rightarrow 0$. In the following, the two quantities will be indicated, for simplicity, with the same symbol (1.14), unless explicitly stated. It should be noticed that the function $P(\varepsilon)$ is not smooth, in general. Its graph is shown in Figure 6, for the case of the Cantor set C (*Devil’s Staircase*). It remains constant along intervals (since there is no mass in the gaps) and makes infinitely many jumps (at each point of the set). The presence of (large) jumps represents a problem from a computational point of view, even though the effect is smeared out when an average over the reference points $\mathbf{x}$ is done. Also, in strange attractors, such a behaviour only occurs along (one or more) stable manifolds and, due to the smoothness of the measure along unstable manifolds, is not very pronounced. If the Cantor set is constructed by keeping at each step 4 intervals of length $1/9$ (equally spaced on the unit segment), the fractal dimension remains the same as for the set of Eq.(1.11) ($D(0) = \ln 4 / \ln 9 = \ln 2 / \ln 3$), but the corresponding Devil’s Staircase displays smaller jumps. The quantity which describes the size of gaps is called *lacunarity* and is connected to the prefactors in the power laws which relate the mass $P(\varepsilon)$ to the ball’s size ε [11]. The average slope of the curve $\ln P(\varepsilon)$ versus $\ln \varepsilon$ yields the dimension, while lacunarity effects manifest themselves as a superimposed oscillation [20,21]. Large gaps correspond to large lacunarity.

In order to give a global estimate of dimension, it is necessary to perform a suitable average over the pointwise dimensions $\alpha(\mathbf{x}_i)$. Accordingly, the generalized (Renyi-) dimensions $D(q)$ can be defined as [22,23,24]

$$D(q) = \frac{1}{q-1} \lim_{\varepsilon \rightarrow 0} \frac{\ln \sum_i P_i^q(\varepsilon)}{\ln \varepsilon}, \text{ for } q \geq 0, \quad (1.15)$$

where a regular grid of boxes of size ε has been taken and $P_i(\varepsilon)$ is the mass in the i -th box. Methods based on Eq. (1.15) are called *box-counting*. For $q = 0$, the sum reduces to the number $N(\varepsilon)$ of non-empty boxes and relation (1.12) is recovered (up to the infimum). For $q \rightarrow 1$, an expansion shows that $D(1)$ is the ratio between the Shannon information $-\sum_i P_i(\varepsilon) \ln P_i(\varepsilon)$ and $-\ln \varepsilon$: for this reason it is called *information dimension* [25]. The dimension $D(2)$ is called *correlation dimension* [26]. The function $D(q)$ is monotonic non-increasing and characterizes

the moments of the distribution of the fluctuating variable $P_i(\epsilon)$. Large q 's enhance the relative contribution of large masses P (small dimension α). The limit $q \rightarrow \infty$ yields the smallest pointwise dimension on the set (corresponding to the most densely populated region). The Hausdorff dimension D_H lies between $D(1)$ and $D(0)$ ($D(0) \geq D_H \geq D(1)$). An example in which $D_H = D(1)$ can be found in [12]. The dimension function $D(q)$ of uniform fractals is constant.

A multi-dimensional extension of the non-uniform Cantor set allows to obtain an implicit relation for $D(q)$, in terms of contraction rates and probability weights [24]. An E -dimensional (hyper-) cube of size 1 and measure 1 is partitioned into M pieces with size r_j ($j = 1, \dots, M$) and mass p_j ($\sum_{j=1}^M p_j = 1$). At the next stage, every piece is further divided into M subsets, each one with a measure reduced by a factor p_j and size by a factor r_j . This model is strictly self-similar (always the same number M of cubes is taken at each step, with the same collection of rates r_j and weights p_j) and isotropic (the same contraction rate r_j is applied along all directions in space). By covering the set with a grid of boxes of size $\epsilon \ll \min\{r_j, j = 1, \dots, M\}$, the sum in Eq.(1.15) can be rewritten as

$$\sum_{i=1}^{N(\epsilon)} P_i^q(\epsilon) = \sum_{(1)} P_i^q(\epsilon) + \sum_{(2)} P_i^q(\epsilon) + \dots + \sum_{(M)} P_i^q(\epsilon), \quad (1.16)$$

where the symbol (j) indicates the j -th subset of the fractal (at the first step of the Cantor construction). Since $\sum_{(j)} P_i = p_j$ (total mass in the j -th piece), the probabilities can be normalized within each subset by multiplying and dividing the j -th sum by p_j^q . Considering, then, that the j -th subset has the same structure as the whole unit hyper-cube, rescaled by a factor r_j , Eq.(1.16) reduces to

$$\epsilon^{(q-1)D(q)} = \sum_{j=1}^M p_j^q (\epsilon r_j^{-1})^{(q-1)D(q)},$$

where the definition of $D(q)$ (1.15) has been used. By simplifying, the self-similarity relation [24]

$$\sum_{j=1}^M p_j^q r_j^{(1-q)D(q)} = 1 \quad (1.17)$$

is obtained. Equation (1.17) is implicit in $D(q)$ and can be solved explicitly only for the information dimension $D(1)$. In the limit $q \rightarrow 0$, the expression becomes independent of the weights p_j , in agreement with the definition of box-dimension. If the fractal is uniform, the mass scales as $p_j = r_j^D$, with the same pointwise dimension D everywhere, and Eq.(1.17) is solved by $D(q) = D$ (i.e., the dimension function is constant).

More generally, at the first step of the recursive process one can define a *partition function* [27]

$$Z(q, \tau, r) = \sum_{j=1}^M \frac{p_j^q}{r_j^\tau}, \quad (1.18)$$

where r is the maximum of the r_j and the parameter τ is to be determined. At the next level, a new partition function

$$Z(q, \tau, r^2) = \sum_{i,j=1}^M \frac{(p_i p_j)^q}{(r_i r_j)^\tau} \quad (1.19)$$

can be evaluated. An exponential behaviour results then for Z , upon iteration of the process:

$$Z(q, \tau, r^n) = Z^n(q, \tau, r) . \quad (1.20)$$

If the fractal is not strictly self-similar, this relation only holds asymptotically (as an order of magnitude) and not as an exact equality at each step (i.e., $Z(q, \tau, r^n) \sim \bar{Z}^n(q, \tau, r)$, with $\bar{Z} \approx Z(q, \tau, r)$). Eventually, for $n \rightarrow \infty$, this partition function will be of the order unity when $\tau = (q-1)D(q)$ (Eq.(1.17)). Moreover, since $Z(q, \tau, r)$ is monotonic non-increasing in q and non-decreasing in τ , one can argue that a unique function $\tau(q)$ exists such that [27]

$$\lim_{r \rightarrow 0} Z(q, \tau, r) = \begin{cases} \infty & \text{for } \tau > \tau(q) , \\ 0 & \text{for } \tau < \tau(q) . \end{cases} \quad (1.21)$$

This is a generalization of the definition of Hausdorff dimension given above. For $\tau = \tau(q)$, the limit can be either finite, or zero, or infinity: the latter two cases usually correspond to a residual logarithmic dependence of Z on r . In general, the rates r_j and the weights p_j actually used to construct the fractal set are unknown. For the application of Eq.(1.18), a partition with M balls of arbitrary size $\varepsilon_j \leq \varepsilon$ must be chosen and the populations p_j computed. The value of τ is, then, determined by letting ε tend to zero ($M \rightarrow \infty$). If a partition with finite M (and, consequently, finite ε_j 's) yields exactly $Z(q, \tau(q), \varepsilon) = 1$, it is called a *generator*: in that case, the "hidden rule" used to produce the fractal has been guessed. The introduction of the partition function (1.18) anticipates the subject of Part II, where the analogy between fractal measures and thermodynamic ensembles will be treated.

The previous definitions can be reformulated by substituting sums with averages. For any quantity Q_i , defined on each non-empty box of the covering adopted in Eq.(1.15), the average value $\langle Q \rangle$ can be approximated by $\langle Q \rangle = \sum_i P_i Q_i$. Hence, the sum in Eq.(1.15) can be written as $\sum_i P_i^q = \langle P_i^{q-1} \rangle$. Similarly, the partition function (1.18) is given by [27,28]

$$Z(q, \tau, \varepsilon) = \left\langle \frac{P_i^{q-1}}{\varepsilon^\tau} \right\rangle , \quad (1.22)$$

where balls of size $\varepsilon_j \leq \varepsilon$ and mass $P_j = P(\varepsilon_j)$ have been chosen randomly with respect to the natural measure ρ (not to Lebesgue's measure). This expression can be simplified in essentially two complementary ways. In fact, if the result does not depend on the collection $\{\varepsilon_j\}$ (as it is usually the case for physical systems),

it is possible to adopt a covering with balls of fixed radius ε centered at randomly chosen points $\mathbf{x}$. In this case, Eq.(1.22) reduces to [23,24]

$$\langle P(\varepsilon)^{q-1} \rangle = \int d\rho(\mathbf{x}) P(\varepsilon, \mathbf{x})^{q-1} \sim \varepsilon^{\tau(q)}, \text{ for } \varepsilon \rightarrow 0, \quad (1.23)$$

where $P(\varepsilon, \mathbf{x})$ is given by Eq.(1.13). Alternatively, if the mass is fixed to the value p and the (variable) radii $\varepsilon_j(p)$ are measured, Eq.(1.22) reduces to [29]

$$\langle \varepsilon(p)^{-\tau(q)} \rangle \sim p^{1-q}, \text{ for } p \rightarrow 0. \quad (1.24)$$

The method is directly implemented by taking M reference balls containing k points each and considering an increasing number n of points on the set ($p = k/n$). The radius $\varepsilon_j(n)$ is, then, the distance between the centre $\mathbf{x}_j$ of the ball and its k -th nearest neighbour [29]. In the case of box-counting methods, since boxes will not be necessarily centered on points of the fractal, some of them will contain spuriously small mass: this is the main reason for the restriction $q \geq 0$ in definition (1.15). Equations (1.23) and (1.24) are, instead, valid for all values of q [28,30]. In the limit $q \rightarrow -\infty$, the largest pointwise dimension is found, corresponding to the least populated region of the set [29].

An overall description of strange attractors can be obtained using these methods (or variants of them [31,32]). To perform a more detailed analysis, which takes into account the anisotropy, local properties of the invariant manifolds must be studied (Lyapunov exponents) and more generic coverings employed (e.g., by using ellipsoids, rather than balls). This will be the subject of Sections 1.5 and 1.6.

1.4 Metric Entropies

The discussion so far has concerned the scaling behaviour of a fractal measure ρ as a function of the observational resolution ε . The pointwise dimension $\alpha(\mathbf{x})$ characterizes the information $I(\varepsilon) = -\ln P(\varepsilon, \mathbf{x}) \sim -\alpha(\mathbf{x}) \ln \varepsilon$ contained in a single measurement of the initial condition $\mathbf{x}$, performed with an accuracy ε . The metric entropy, instead, refers to a sequential measurement, i.e. to a series of observations of a trajectory at equally spaced times: $t_i = i\Delta t$, in the case of continuous time and $t_i = i$, for maps ($i = 1, \dots, n$). To be more specific, let us consider the discrete-time dynamical system (1.2). The mapping is strictly deterministic: given the initial condition $\mathbf{x}_0$ with infinite precision, the trajectory $\mathbf{x}_i = \mathbf{F}^i(\mathbf{x}_0)$ is uniquely determined. Let us also assume that an observation of the system be done with limited resolution: i.e., the phase-space A is partitioned in M disjoint regions B_j (of size ε_j), such that $A = \cup_j B_j$. When the representative point $\mathbf{x}_i$ is in the element B_j , the “reading instrument” displays the value j . In this way, a symbolic sequence $S_n = \{s_i, i = 1, \dots, n\}$ can be associated to each trajectory $\{\mathbf{x}_i, i = 1, \dots, n\}$ (for finite n , S_n is also called *word*) [33]. The symbol $s_i \in \{1, M\}$ represents the index of the partition element visited by the trajectory at time i . The order of symbols in the word $\{s_i\}$ is now relevant, at variance with the case of fractal dimensions where different subsets of equal size and mass (characterized by permutations of the same sequence) were all equivalent for the application of Eq.(1.18). In other words, various dynamical systems can admit the same invariant measure, while exhibiting quite different time evolution.

Given an initial condition $\mathbf{x}_0$ in element B_n , the image $\mathbf{F}^n B_n$ at time n will be quite extended, due to the exponential divergency of nearby orbits. Hence, $\mathbf{F}^n B_n$ will, in general, intersect more than one set B_j , indicating that the uncertainty on the initial condition has been amplified by the time evolution. The conditional probability of being in subset B_k at time n , given an initial condition in B_j is [34]

$$P(k, n | j) = \rho(\mathbf{F}^n B_j \cap B_k) / \rho(B_j), \quad (1.25)$$

where ρ is the invariant measure (see Figure 7). The mass $\rho(B_j)$ is the same as that contained in $\mathbf{F}^n B_j$, and the conditional probability is normalized in such a way that

$$P(k, 0 | j) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{if } j \neq k. \end{cases} \quad (1.26)$$

For a *mizing* system [12,35],

$$\lim_{n \rightarrow \infty} P(k, n | j) = \rho(B_k), \quad (1.27)$$

i.e. the image of the initial set B_j covers the whole attractor and no correlation between final and starting element survives. The metric entropy can be seen as the rate at which the mass in a starting box spreads over the attractor and, thus, can be related to the difficulty of predicting the future time evolution of the system. Alternatively, one can consider it as a rate of creation of information [1]: two

initial conditions that are closer than the observational resolution ϵ will evolve into distinct states, after some time. Due to this, a sequence of measurements can be used to localize the initial condition with great precision. In fact, if the symbolic

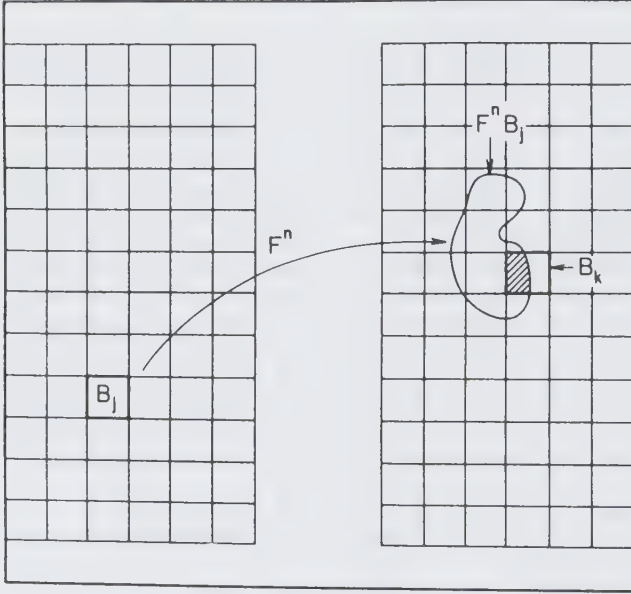


Figure 7: Illustration of the spread of an initial set B_j over the attractor in n iterates.

sequence begins with $\{j, k, \dots\}$, the system is in element B_j at time $i = 0$ and in B_k at time $i = 1$: hence, it was somewhere inside $\mathbf{F}^{-1}B_k$ at time $i = 0$. Thus, the only initial conditions which could produce the sequence $\{j, k, \dots\}$ had to belong to $B_j \cap \mathbf{F}^{-1}B_k$ (which is a subset of B_j), at $i = 0$: since the set of admissible initial conditions has been restricted by the joint observation at times $i = 0, 1$, some information has been gained with respect to the sole knowledge $\mathbf{x}_0 \in B_j$. A word S_n , composed of n symbols, can be assigned to smaller and smaller sets, for increasing n , since fewer and fewer initial conditions will generate orbits following the same path $B_{s_1} \rightarrow B_{s_2} \rightarrow \dots B_{s_n}$, without escaping to other elements [33]:

$$\mathbf{F}^i(\mathbf{x}_0) \in B_{s_i}, \forall i = 1, \dots, n \Leftrightarrow \mathbf{x}_0 \in \bigcap_{i=1}^n \mathbf{F}^{-i}B_{s_i}. \quad (1.28)$$

If an infinite sequence ($n \rightarrow \infty$) individuates single points (and not small, but finite, regions), the partition is called *generating*: it attributes unambiguously a unique initial condition to every symbol sequence S_∞ . A word is admissible if it corresponds to at least one point (on which the natural measure is non-zero: unstable periodic orbits generate ‘special’ sequences). The larger the number of allowed words, the more complex the system is. For this reason, in analogy with the concept of fractal dimension (1.12), the degree of complexity of a dynamical system can be estimated from the asymptotic behaviour of the number of possible sequences of length n , when $n \rightarrow \infty$. The intersection between two partitions $\mathcal{A} = \{A_l, l \in [1, L]\}$ and $\mathcal{B} = \{B_m, m \in [1, M]\}$, is defined as [12,36]: $\mathcal{A} \vee \mathcal{B} = \{A_l \cap B_m, \forall l, m\}$. If $\mathcal{A}$ is the set of the inverse images of the elements of $\mathcal{B}$, the intersection $\mathcal{A} \vee \mathcal{B}$ is called a *refinement* of $\mathcal{B}$. Let us consider the n -th refinement of $\mathcal{B}$:

$$\mathcal{B}^n = \bigvee_{i=0}^{n-1} \mathbf{F}^{-i} \mathcal{B}. \quad (1.29)$$

Each non-empty element of $\mathcal{B}^n$ contains initial conditions which all produce the same word of length n . Indicating with $N(\mathcal{B}^n)$ the minimum number of admissible words (i.e., of non-empty subsets of $\mathcal{B}^n$), the *topological entropy* of $\mathbf{F}$ with respect to the covering $\mathcal{B}$ is defined as [12,33,36]

$$h(\mathbf{F} | \mathcal{B}) = \lim_{n \rightarrow \infty} \frac{\ln N(\mathcal{B}^n)}{n}. \quad (1.30)$$

That is, the number of new elements visited by points in the original subset grows exponentially in time. The topological entropy of the mapping $\mathbf{F}$ is, then,

$$h(\mathbf{F}) = \sup_{\mathcal{B}} h(\mathbf{F} | \mathcal{B}). \quad (1.31)$$

The supremum is taken over all possible coverings $\mathcal{B}$ and it is achieved for those partitions $\mathcal{B}$ which are generating: in that case, $h(\mathbf{F}) = h(\mathbf{F} | \mathcal{B})$ [12,36]. Any refinement $\mathcal{B}^n$ of a generating partition is, of course, still generating and becomes infinitely sharp, when $n \rightarrow \infty$. The solutions of a dynamical system are chaotic if $h > 0$ (regular motion corresponds to $h = 0$ and pure stochasticity to $h = +\infty$).

When a generating partition can be found, the study of the symbol sequences S_n (called *symbolic dynamics*) is equivalent to the analysis of the real trajectories, for the characterization of the dynamical complexity of the motion. The action of the map $\mathbf{F}$ in phase-space is translated into a *shift* of symbols in the associated space:

$$\mathbf{x}_i \mapsto \mathbf{x}_{i+1} \Rightarrow \{\dots, s_{i-1}, s_i, s_{i+1} \dots\} \mapsto \{\dots, s_i, s_{i+1}, s_{i+2}, \dots\}, \quad (1.32)$$

that is, the time origin in the (doubly-infinite) symbol sequence is moved one place to the right. In the case of the logistic map (1.9), the (binary) generating partition is obtained by cutting the x -axis at $x = 0$. A trajectory is thus mapped onto a sequence of 0’s (for $x < 0$) and 1’s (for $x > 0$). Neither are all sequences

admissible, nor are they equally probable, except when the control parameter a is equal to 2. For generic systems, there are no simple methods to individuate generating partitions, especially if composed of a few subsets [37]. However, it is easy to find many of them which are composed of a large number of elements [38].

The topological entropy is the dynamical analogue of the box dimension (1.12) and, in the same way, does not depend on the frequency of occurrence of each symbol sequence. Therefore, similarly to what has been done in Section 1.3 for the generalized dimensions, the probability distribution of words in orbit-space must be now considered:

$$P(s_0, \dots, s_{n-1}) = \rho(B_{s_0, \dots, s_{n-1}}) = \rho(B_{s_0} \cap \mathbf{F}^{-1}B_{s_1} \cap \dots \cap \mathbf{F}^{-n+1}B_{s_{n-1}}), \quad (1.33)$$

where ρ is the invariant measure. Clearly, the probability $P(s_0, \dots, s_{n-1})$ to observe the trajectory $\{s_0, s_1, \dots, s_{n-1}\}$ is just given by the mass contained in the set $B_{s_0, s_1, \dots, s_{n-1}}$ of initial conditions which generate that sub-sequence of length n . A set of orbits whose first n symbols coincide is called n -cylinder [39]. The generalized metric entropies $h(q)$ are defined as

$$h(q) = \sup_{\mathcal{B}} \lim_{n \rightarrow \infty} \frac{1}{n} \frac{1}{1-q} \ln \sum_{S_n} P^q(S_n), \quad (1.34)$$

where the sum extends over all allowed sequences S_n . For $q \rightarrow 0$, the definition of topological entropy is recovered, while $h(1)$ is the *metric entropy* in the sense of Kolmogorov [12,33,36]. When a generating partition is not known, as in the case of experimental data or in most of computer simulations, it is possible to divide the space in boxes (or balls) of size $\varepsilon_j \leq \varepsilon$ (although they are usually taken all equal, for simplicity) and evaluate the metric entropies as

$$K(q) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \frac{1}{1-q} \ln \sum_{S_n} P^q(S_n), \quad (1.35)$$

where the supremum is now obtained by letting ε tend to 0: in case a generating partition has been found, the limit is not necessary. The probability $P(S_n)$ in Eq.(1.35) is defined exactly as in Eq.(1.34): the only difference is in the choice of the partition. However, $P(S_n)$ can be interpreted (and computed) in an alternative way. Let us *embed* the orbit $\{\mathbf{x}_i\}$ in an embedding space of dimension $E \times n$, the points of which are defined as

$$X_i(n) \equiv (\mathbf{x}_i, \mathbf{x}_{i+1}, \dots, \mathbf{x}_{i+n-1}), \quad (1.36)$$

where the argument n indicates the length of the orbit chosen. Then, all points $Y_j(n)$ lying within a distance ε from $X_i(n)$ in the embedding space correspond to an n -cylinder of radius ε in the space of trajectories. Therefore, the probability $P(\varepsilon, X_i(n))$ to find a generic embedded point $Y_j(n)$ in an ε -neighbourhood of $X_i(n)$ is also the probability to observe the segment of trajectory (1.36), within an accuracy ε . It is then possible to rewrite Eq.(1.35) as an average over reference points

(balls) in the embedding space [40,41]:

$$K(q) = \lim_{\varepsilon \rightarrow 0} \lim_{n \rightarrow \infty} \frac{1}{n} \frac{1}{1-q} \ln \langle P^{q-1}(\varepsilon, X_i(n)) \rangle, \quad (1.37)$$

where the weight is given by $P(\varepsilon, X_i(n))$ itself. If the metric of the maximum

$$d(X_i(n), Y_j(n)) = \max_{k \in [0, n-1]} \|x_{i+k} - y_{j+k}\| \quad (1.38)$$

is adopted, $P(S_n)$ is just the mass contained in some ellipsoid around x_0 , whose points originate trajectories remaining within a distance ε from the actual trajectory x_k , for all times $k \in [0, n-1]$ [42,43] (see Figure 8, where an illustration for the continuous-time case is presented). The symbol $\|\cdot\|$ indicates the euclidean norm in R^n .

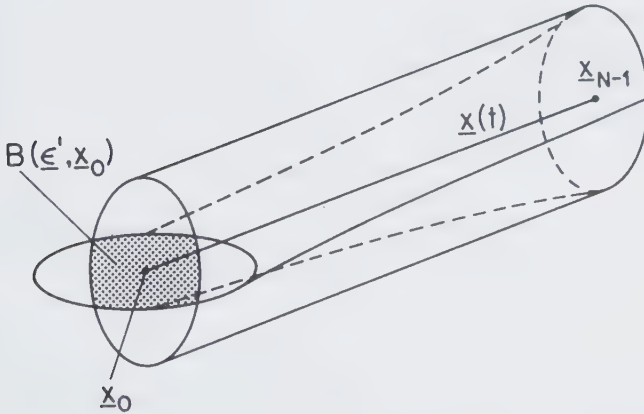


Figure 8: All trajectories flowing out of the region $B(\varepsilon', x_0)$ remain for a time $n\Delta t$ within a distance $\varepsilon = \varepsilon'_2$ (the radius of the cylinder) from the actual trajectory $x(t)$.

Thus, the metric entropy can be seen to be related to a fractal dimension in the space of trajectories [33,34]: the latter is covered with cylinders of spatial extension ε and time length n (the analogous of ε along the time direction is ε^{-n}).

Both fractal dimensions $D(q)$ and metric entropies $K(q)$ are invariant under smooth coordinate changes (i.e., independent of the choice of coordinates). The metric entropies $K(q)$ coincide with the previously defined $h(q)$ and satisfy: $K(q_1) \geq K(q_2)$, for $q_2 > q_1$.

1.5 Lyapunov Exponents

As already discussed in Sections 1.1 and 1.4, the motion on a strange attractor is unpredictable due to the exponential divergency of nearby orbits. The quantities which characterize this phenomenon are the Lyapunov exponents [13].

In order to define them, let us consider an ε -ball $B(\varepsilon, \mathbf{x}_0)$, centered at $\mathbf{x}_0$. In a time n , it evolves into an ellipsoid with semi-axes

$$\varepsilon_k \approx \varepsilon \cdot \mu_k(\mathbf{x}_0, n) \equiv \varepsilon \cdot e^{n\lambda_k(\mathbf{x}_0, n)}, \text{ for } \varepsilon \rightarrow 0, \quad (1.39)$$

where $\mu_k(\mathbf{x}_0, n)$ is the modulus of the *multiplier* (stretching factor) along the k -th invariant manifold ($k \in [1, E]$), computed over the finite portion of trajectory $X_0(n)$ (see Eq.(1.36)), and $\lambda_k(\mathbf{x}_0, n)$ is the k -th *effective Lyapunov exponent* [30,43,44] corresponding to the same piece of orbit. For $n \rightarrow \infty$, the effective Lyapunov exponents converge to the average Lyapunov exponents $\langle \lambda_k \rangle$ (provided that they refer to an initial condition $\mathbf{x}_0$ with non-zero measure ρ). In the following, they will be assumed to be ordered by magnitude: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_E$. The motion is chaotic if at least one average Lyapunov exponent is positive.

Equivalently, Lyapunov exponents can be defined through the matrix $D\mathbf{F}^n(\mathbf{x}_0)$ of partial derivatives of the n -th iterate of the map (1.2), with initial condition $\mathbf{x}_0$. Let us consider the limit (which exists for ρ -almost all $\mathbf{x}_0$) [1,45]

$$T(\mathbf{x}_0) = \lim_{n \rightarrow \infty} [\overline{D\mathbf{F}^n}(\mathbf{x}_0) D\mathbf{F}^n(\mathbf{x}_0)]^{1/(2n)}, \quad (1.40)$$

where the bar denotes the adjoint matrix. Indicating with $E_k(\mathbf{x}_0)$ the subspace of R^E corresponding to the eigenvalues of $T(\mathbf{x}_0)$ which are $\leq \langle e^{\lambda_k} \rangle$, then $R^E = E_1(\mathbf{x}_0) \supset E_2(\mathbf{x}_0) \supset \dots \supset E_E(\mathbf{x}_0)$ and

$$\langle \lambda_k \rangle = \lim_{n \rightarrow \infty} \frac{1}{n} \ln \|D\mathbf{F}^n(\mathbf{x}_0) \mathbf{v}_k\|, \quad (1.41)$$

if $\mathbf{v}_k \in E_k(\mathbf{x}_0) \setminus E_{k+1}(\mathbf{x}_0)$. Almost all vectors $\mathbf{v}$ which are not in $E_2(\mathbf{x}_0)$ yield the largest Lyapunov exponent $\langle \lambda_1 \rangle$. The Lyapunov exponents generalize the concept of eigenvalues at a fixed point (or limit cycle) to give average expansion and contraction rates along a trajectory, in the linear approximation. The subspace $E_1 \setminus E_2$ consists of all vectors $\mathbf{v}$ in the tangent space at $\mathbf{F}^n$ which grow at the fastest possible rate, while vectors in $E_2 \setminus E_3$ grow at the next fastest rate, and so on [16]. Infinitesimal vectors in R^E diverge exponentially according to the first multiplier $\mu_1(n) = e^{n\lambda_1}$, in a time n ; elements of surface grow like $\mu_1(n)\mu_2(n) = e^{n(\lambda_1+\lambda_2)}$. Hence, by considering elements of ‘volume’ of increasing dimension, since the system is dissipative, there will be an index j such that $\sum_{k=1}^j \lambda_k \geq 0$ but $\sum_{k=1}^{j+1} \lambda_k < 0$. That is, j -dimensional volumes are still expanded (or conserved), while higher-dimensional volumes are contracted. Therefore, the dimension of the attractor will be larger or equal to j . A precise relation between dimension and Lyapunov exponents will be derived in the next section. For the flow (1.1), the sum of all Lyapunov exponents equals the divergence $\nabla \cdot \mathbf{F}$ (if this is position-independent)

and, for maps with constant jacobian J , it coincides with $\ln |J|$. In the case of flows, at least one Lyapunov exponent is equal to zero: it corresponds to the direction tangent to the trajectory (distances remain constant, as an average, along the orbit). In one dimensional maps, the average Lyapunov exponent is given by $\langle \lambda \rangle = \langle \ln |F'(x)| \rangle$. A map is *hyperbolic* if its attractor has no Lyapunov exponents equal to zero (there is an expanding direction at each point); the same is true for a flow, once the direction tangent to the orbit is excluded (see [1] for a more precise definition). The two approaches (1.39) (ellipsoids) and (1.41) (products of Jacobi matrices $D\mathbf{F}^n = \prod_{i=0}^{n-1} D\mathbf{F}(x_i)$) are equivalent for hyperbolic systems. In the non-hyperbolic case, the stable and unstable manifolds of an unstable fixed point P (Fig. 3) may exhibit quadratic tangencies (*homoclinic* tangencies) to each other in some points Q_i [16], at variance with hyperbolic systems where only transversal intersections occur. In such a case, the stable and unstable directions are essentially indistinguishable one from the other, in every small region around the tangency points Q_i : therefore, effective Lyapunov exponents evaluated from the linearized matrix (Eq.(1.41)) might not be ordered in magnitude, when referring to a piece of trajectory which remains close to some tangencies Q_i , for a certain number of iterates.

Examples of one-dimensional hyperbolic maps are the asymmetric Bernoulli shift

$$y_{n+1} = \begin{cases} y_n/p_1 & \text{if } y_n \leq p_1, \\ (y_n - p_1)/p_2 & \text{if } y_n > p_1 \end{cases} \quad (1.42)$$

and the tent map

$$y_{n+1} = \begin{cases} y_n/p_1 & \text{if } y_n \leq p_1, \\ (1 - y_n)/p_2 & \text{if } y_n > p_1, \end{cases} \quad (1.43)$$

where $p_1 + p_2 = 1$. At each iteration, the multiplier can assume either the value $\mu = 1/p_1$, if $y_n \leq p_1$, or $\mu = \pm 1/p_2$, if $y_n > p_1$ (the plus sign refers to the map (1.42) and the minus sign to Eq.(1.43)): distances are always stretched. In the case of the logistic map (1.9), instead, the slope $|F'(x)|$ is smaller than 1 in a region around the maximum at $x = 0$. Therefore, the multiplier $\mu(n) = \prod_{i=1}^n |F'(x_i)|$ over n iterates can be smaller than 1, even for very large values of n , if one or more points of the trajectory pass close to the maximum. The probability distribution for the effective Lyapunov exponents of this map will be studied in Part III, in the limit $n \rightarrow \infty$. An example of a two-dimensional non-hyperbolic transformation is provided by the Hénon map (1.3), for which numerical studies [37,30] and mathematical conjectures [46,16] suggest the actual occurrence of homoclinic tangencies. A piece of orbit passing in the neighbourhood of such points will exhibit a small (and, possibly, negative) first effective Lyapunov exponent λ_1 . Since the sum $\lambda_1 + \lambda_2 = \ln |J| < 0$ is constant, both λ_1 and λ_2 could be negative, or the second effective Lyapunov exponent larger than the first one, when evaluated from Eq.(1.41). As in the case of the logistic map, this situation can persist for a large number of iterates n (with decreasing probability for increasing n).

Since the Lyapunov exponents (or, more precisely, the multipliers μ_k) represent the stretching factors of infinitesimal volumes along each direction in phase space, they can be related both to metric entropies (degree of chaoticity of the system) and to fractal dimensions (local scaling properties of probability densities, under successive magnifications). These connections will be studied in the next two sections.

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1.6 Dimensions and Lyapunov Exponents

The relation between fractal dimensions and Lyapunov exponents can be discussed first on a simple model of a hyperbolic strange attractor, which is analytically solvable: the generalized baker transformation [17], defined by

$$(x_{n+1}, y_{n+1}) = \begin{cases} (r_1 \cdot x_n, y_n/p_1) & \text{if } y_n \leq p_1, \\ (r_2 \cdot x_n + 1 - r_2, (y_n - p_1)/p_2) & \text{if } y_n > p_1, \end{cases} \quad (1.44)$$

where $r_1 + r_2 < 1$ and the equation for y is again the asymmetric Bernoulli shift (1.42). The action of the map can be described as follows: the unit square is cut horizontally at a height $y = p_1$ and the two parts are contracted in the x -direction according to the single-step multiplier $\mu_2(1)$, which can assume either the value r_1 or r_2 , depending on y_n , with probabilities p_1 and p_2 , respectively. The two resulting rectangles are then stretched along the y -direction by the factor $\mu_1(1)$ (first multiplier computed over one iteration), equal to $1/p_1$ in the lower part and to $1/p_2$ in the upper one, to retrieve length one. Since the attractor must be bounded, the upper rectangle is, finally, placed back into the rightmost part of the unit square and the whole procedure is iterated. The process of construction of the fractal can thus be interpreted from a dynamical point of view as the repeated application of expansion and contraction rates onto a set of initial conditions (the unit square, in this case). For what concerns symbol sequences, a generating partition is obtained by cutting the square horizontally at the height p_1 : the bit 0 corresponds to $y_n \leq p_1$ and the bit 1 to $y_n > p_1$. The asymptotic attractor is the product of a continuum, along the y -axis, by a Cantor set, along the x -axis. To take into account this anisotropy, it is useful to consider *partial* dimensions [47,28] $d_k(q)$ ($k = 1, \dots, E$), along each invariant manifold: in the case of the baker map, $d_1(q) = 1$ for all q , since the invariant measure of the Bernoulli shift (1.42) is constant, and $d_2(q)$ is the solution $D(q)$ of Eq.(1.17), for $M = 2$, since a horizontal section of the attractor coincides with the nonuniform Cantor set discussed in Section 1.3. The overall dimension of the attractor is given by the sum $d_1(q) + d_2(q)$. More generally, to characterize the scaling properties of non-isotropic fractals (also called *self-affine* [11]), it is necessary to adopt coverings consisting of collections of ellipsoids, the axes of which are allowed to scale with different speed, rather than using balls or boxes, as in Section 1.3.

Restricting the analysis to fractals which are locally a direct product of continua and Cantor sets (non-hyperbolic systems are excluded, due to the homoclinic tangencies), consider a covering with ellipsoids $B(\vec{\epsilon}, \mathbf{x})$ having axes $\vec{\epsilon} \equiv (\epsilon_1, \dots, \epsilon_E)$, directed along each local invariant manifold. The mass $P(\vec{\epsilon}, \mathbf{x})$ contained in $B(\vec{\epsilon}, \mathbf{x})$ factorizes into the product of probabilities along the local coordinate axes and scales as

$$P(\vec{\epsilon}, \mathbf{x}) \sim \prod_{k=1}^E \epsilon_k^{\alpha_k(\mathbf{x})}, \quad (1.45)$$

for $\vec{\varepsilon} \rightarrow 0$, with $\alpha_k(\mathbf{x})$ denoting the pointwise partial dimension along the k -th local invariant manifold [30]. As already discussed in Section 1.3, rather than studying local quantities like $\alpha_k(\mathbf{x})$, a global approach can be followed by averaging Eq.(1.45) over the invariant measure $\rho(\mathbf{x})$. Hence, a generalization of Eq.(1.23) is obtained:

$$\langle P(\vec{\varepsilon})^{q-1} \rangle \sim \prod_{k=1}^E \varepsilon_k^{\tau_k(q)}, \text{ for } \vec{\varepsilon} \rightarrow 0, \quad (1.46)$$

where

$$\tau_k(q) = (q-1)d_k(q). \quad (1.47)$$

The partial dimensions $d_k(q)$ are bounded between 0 and 1, for positive q , and their sum is the overall dimension $D(q)$ of the set [1,47,43,28,30]:

$$0 \leq d_k(q) \leq 1, \text{ for } q \geq 0, \quad (1.48)$$

$$\sum_{k=1}^E d_k(q) = D(q). \quad (1.49)$$

In the following, the vector notation $\vec{\tau}(q)$, $\vec{d}(q)$, $\vec{\lambda}$ will be used, for space reasons. If ellipsoids with position-dependent size are chosen, a more general relation than (1.46) holds (in analogy with Eq.(1.22)):

$$\left\langle P(\vec{\varepsilon})^{q-1} / \prod_{k=1}^E \varepsilon_k^{\tau_k(q)} \right\rangle \sim 1. \quad (1.50)$$

Consider, then, the effect of the dynamical law (1.1) or (1.2) onto the elements of the covering (the evolution is assumed to be invertible as for the baker map (1.44) or the Hénon map (1.3), but not for one-dimensional transformations like (1.9), (1.42) and (1.43)). After n iterations, an ellipsoid with axes ε_k , centered at $\mathbf{x}_0$, will be transformed into an ellipsoid with the same mass $P(\vec{\varepsilon})$ but with axes $\varepsilon'_k \approx \varepsilon_k \cdot \mu_k(\mathbf{x}_0, n) \equiv \varepsilon_k \cdot e^{n\lambda_k(\mathbf{x}_0, n)}$ (see Eq.(1.39)). Substituting the expression for ε'_k into Eq.(1.50), the invariance of the measure implies that [30]

$$\left\langle \frac{P(\vec{\varepsilon})^{q-1}}{\prod_k \varepsilon_k^{\tau_k(q)}} \cdot e^{-n\vec{\lambda} \cdot \vec{\tau}(q)} \right\rangle \sim 1, \text{ for } \vec{\varepsilon} \rightarrow 0 \text{ and } n \rightarrow \infty. \quad (1.51)$$

This relation represents a constraint on the partial dimensions $d_k(q)$ and constitutes, essentially, a statement about the joint distribution of the pointwise crowding indices $\alpha_k(\mathbf{x}_0, \vec{\varepsilon})$, implicitly present in $P(\vec{\varepsilon})$ (see Eq.(1.45)), and effective Lyapunov exponents (evaluated over trajectories starting from the reference ellipsoids centered on $\mathbf{x}_0$). The global partial dimensions $d_k(q)$ must be chosen in such a way as to balance the decrease of the mass $P(\vec{\varepsilon})$, with decreasing $\vec{\varepsilon}$, and keep stationary the generalized volume-expansion rate $e^{n\vec{\lambda} \cdot \vec{d}(q)}$, for $n \rightarrow \infty$, on the average. Assuming that, for large n , spatial ($\alpha_k(\mathbf{x}_0, \vec{\varepsilon})$) and dynamic ($\lambda_k(\mathbf{x}_0, n)$) terms become independent, the average (1.51) factorizes and, recalling (1.50), the following relation is obtained [42,48,30]:

$$\lim_{n \rightarrow \infty} \left\langle e^{-n\vec{\lambda} \cdot \vec{\tau}(q)} \right\rangle = 1. \quad (1.52)$$

The factorization assumption holds for hyperbolic attractors and its limitations will be discussed in Section 1.8, in connection with strange repellers.

Relation (1.52) can be interpreted as a conservation law for the moments of order $1 - q$ of the generalized volume-expansion rate $V(q, n) = e^{n\tilde{\lambda} \cdot \vec{d}(q)} = \prod_{k=1}^E \mu_k(n)^{d_k(q)}$, where a special metric has been chosen which assigns the global partial dimension $d_k(q)$ to the k -th invariant direction (characterized by the multiplier μ_k). Of course, the single condition (1.52) cannot determine univocally the E unknowns $d_k(q)$; however, it can be used to yield upper bounds to the dimension function $D(q)$. In fact, starting with the largest possible value for $D(q)$ (i.e., $d_k(q) = 1, \forall k$), it is easily seen that the expression on the l.h.s. of Eq.(1.52) vanishes (the ordinary volume shrinks to zero in time), for $q < 1$. Therefore, a smaller value for $D(q)$ must be chosen to meet the requirement (1.52). The smallest possible decrement is accomplished via a gradual decrease of the dimensions $d_k(q)$, starting from the last one. If, however, the “last” $d_k(q)$ ’s (those with high index k , corresponding to negative Lyapunov exponents) are chosen too small, the average in (1.52) diverges, for $q < 1$ (the contrary happens for $q > 1$). As a result, defining the index j_q as the largest integer for which

$$\lim_{n \rightarrow \infty} \left\langle \prod_{k=1}^{j_q} e^{n(1-q)\lambda_k(n)} \right\rangle^{1/n(1-q)} \geq 1, \quad (1.53)$$

the upper bound for $D(q)$ corresponds to $d_k(q) = 1$, for $k \leq j_q$, $d_k(q) = 0$, for $k > j_q + 1$, and $d_{j_q+1}(q)$ determined by relation (1.52) [48,49]. To understand the meaning of this result, consider the limit $q \rightarrow 1$ of Eq.(1.52):

$$\sum_{k=1}^E d_k(1) \langle \lambda_k \rangle = 0. \quad (1.54)$$

This condition has been used in Ref. [50] to give an upper bound to the information dimension $D(1)$ as follows: $D(1) \leq D_L = j_1 + \sum_{k=1}^{j_1} \langle \lambda_k \rangle / |\langle \lambda_{j_1+1} \rangle|$, where D_L is called *Lyapunov dimension* and j_1 is the largest integer for which the sum $\sum_{k=1}^{j_1} \langle \lambda_k \rangle$ is non-negative. This choice is equivalent to assuming that the attractor is continuous along the first j_1 directions and has a Cantor structure along the $(j_1 + 1)$ -th direction only. Indeed, for hyperbolic systems, the partial pointwise dimensions along the unstable directions are equal to 1. On the other hand, systems could be constructed for which more than one contracting direction is Cantor-like [51]. The index j_q represents the integer part of the (upper bound to the) dimension function $D(q)$ and may change with q , indicating that the number of relevant dimensions of the system depends on the order of the moment one is interested in. Such a case will be discussed, in part III, within the framework of the thermodynamic formalism for dynamical systems. It has been proven [52] that the equality $D(1) = D_L$ holds for generic (i.e., not necessarily hyperbolic) two-dimensional diffeomorphisms (see also [1]). Relation (1.52), although implicit in the unknowns $d_k(q)$, can be used to yield very precise numerical estimates of $D(q)$ and, for many

systems with constant jacobian, it can be rendered explicit for d_{j_q+1} , under the assumption that only one Cantor direction exists [48].

The conservation law (1.51) is an extension of Eq.(1.50), since it takes into account the dynamical properties of the system, by requiring the invariance of the (static) partition function $Z(q, \tau, \varepsilon)$ (1.22) under time evolution. The Cantor structure described by $Z(q, \tau, \varepsilon)$ at time n has been determined by expansions and contractions over a set of initial conditions in phase-space, which have been applied during the first n iterates. For example, the width $\delta(m)$ of a generic vertical strip of the baker attractor is given by $\delta(m) = r_1^i r_2^{m-i} = \mu_2(m)$ (second multiplier computed over m steps). The index i denotes the number of times the bit 0 occurred in the symbol sequence. The mass contained in such a strip is the same as that contained in its m -th pre-image (a horizontal strip of height $p_1^i p_2^{m-i}$) and is given by the inverse of the first multiplier $\mu_1^{-1}(m) = p_1^i p_2^{m-i}$, which also represents the probability of the associated symbol sequence. Relation (1.51) expresses the persistence of the average scaling behaviour (1.50) of the mass along the invariant directions during the next n ($n \rightarrow \infty$) steps. Indeed, the Lyapunov exponents in Eq.(1.51) are computed over n forward iterates of the reference point $\mathbf{x}_0$, while the pointwise dimensions $\alpha_k(\mathbf{x}_0)$ in the same point are constant (equal to 1) along the expanding directions and depend on the previous history along the contracting directions. Therefore, for $n \rightarrow \infty$, the factorization assumption is expected to hold, for hyperbolic systems. For non-hyperbolic systems, some difficulties arise. First of all, the assumption (1.45) is not justified, in the neighbourhood of homoclinic tangencies. However, in performing averages like (1.50) or (1.51), the relative weight of such regions is negligible, with respect to the weight of generic (i.e., hyperbolic) points. Secondly, since along unstable manifolds there can be an alternation of expansion and contraction (and not only expansion, as in the hyperbolic case), the corresponding multiplier may be smaller than 1, for finite n , and can no longer be identified with the inverse of a probability, as before. These points will be further discussed in terms of local quantities in Section 1.8.

Finally, notice that successive values of the single-step multipliers $\mu_k(1)$ are completely uncorrelated from each other for the baker map. In fact, the symbol sequence $\{s_i\}$ (the same as for the Bernoulli shift (1.42)) is a string of random digits 0 and 1, chosen with probability p_1 and p_2 , respectively, which satisfy

$$\langle s_i \rangle = p_2, \text{ and } \langle s_i s_j \rangle = \begin{cases} p_2 & \text{if } i = j, \\ p_2^2 & \text{if } i \neq j. \end{cases}$$

As a consequence, relations (1.51) and (1.52) hold, in such a case, for any value of n . For a generic system (hyperbolic or not), the pointwise values of the multipliers are correlated and only the large- n behaviour of $\mu(n)$ is relevant. The presence of such correlations will play a crucial role in parts II and III, when time will be interpreted as a one-dimensional lattice and the symbol s_i as a discrete spin value at lattice-site i .

1.7 Metric Entropies and Lyapunov Exponents

The relation between metric entropies and Lyapunov exponents can be derived by considering the probability $P(\vec{\varepsilon}, X_0(n))$ to find a trajectory which remains for a time n within a cylinder of elliptic cross-section $\vec{\varepsilon}$, around the reference orbit (point in the embedding space) $X_0(n)$. The axes of the ellipsoids are oriented, at each time $i \in [0, n-1]$, along the local invariant manifolds. In analogy with the definition of pointwise dimension (1.14), it is possible to assign a *pointwise entropy* $\kappa(\mathbf{x}_0, n)$ to the trajectory $X_0(n)$, through [30]

$$\kappa(\mathbf{x}_0, n) = \lim_{n \rightarrow \infty} -\frac{1}{n} \ln P(\vec{\varepsilon}, X_0(n)). \quad (1.55)$$

The same quantity without the limit $n \rightarrow \infty$ is the crowding index in trajectory space. In Eq.(1.55), it is assumed that the result becomes independent of the choice of the ε_k 's, when they are small enough. Since, for $n = 0$, $P(\vec{\varepsilon}, X_0(n))$ reduces to the probability $P(\vec{\varepsilon}, \mathbf{x}_0)$ to find a point in an ellipsoid of axes $\vec{\varepsilon}$ around $\mathbf{x}_0$, from Eqs.(1.45) and (1.55), it could be argued that

$$P(\vec{\varepsilon}, X_0(n)) \sim \prod_{k=1}^E \varepsilon_k^{\alpha_k(\mathbf{x}_0)} \cdot e^{-n\kappa(\mathbf{x}_0, n)}, \text{ for } \vec{\varepsilon} \rightarrow 0, n \rightarrow \infty. \quad (1.56)$$

That is, the probability to be in the cylinder factorizes into the probability to be in the initial ellipsoid, at time 0, times the conditional probability to remain within the cylinder for other $n-1$ iterates. However, this assumption needs to be clarified. In fact, for the determination of $\kappa(\mathbf{x}_0, n)$ from Eq.(1.55) or (1.56), it is possible to fix a certain $\vec{\varepsilon}$ and measure the probability $P(\vec{\varepsilon}, X_0(n))$ as a function of n : the procedure is, then, repeated for smaller and smaller $\vec{\varepsilon}$, until convergence to the asymptotic value is reached. Viceversa, fixing $n > 1$ and determining $\alpha_k(\mathbf{x}_0)$ from the measured mass $P(\vec{\varepsilon}, X_0(n))$, as a function of the ε_k 's, does not necessarily yield the right pointwise dimension at $\mathbf{x}_0$, since $P(\vec{\varepsilon}, X_0(n))$ is the mass in the whole cylinder around $X_0(n)$ and the crowding index might not be constant along the orbit $X_0(n)$ (different orbits in the cylinder, in fact, diverge from the central one). In other words, κ characterizes the entire trajectory while the α_k 's refer to a point on the orbit. However, if n is not too large (depending on $\vec{\varepsilon}$), the measured α_k 's can be considered constant in the cylinder: the smaller the ε_k 's, the larger n is allowed to be. In order to study the relation between the probability distribution of the local entropies and that of the local dimensions, it will be necessary to fix the dependence of n on $\vec{\varepsilon}$ (or viceversa). These difficulties are particularly evident for non-hyperbolic systems. Consider the trajectory corresponding to the iterates of the maximum of the logistic map (1.9), for $a = 1.85$, and all other orbits whose first n iterates remain within a distance ε from it. The invariant probability measure, displayed in Fig. 4, presents square-root singularities at the forward images of the maximum and, therefore, a pointwise dimension $\alpha = 1/2$, in those points ($P(\varepsilon) \sim \varepsilon^{1/2}$). However, points belonging to the near trajectories

become more and more distant from the central orbit and may exhibit a different crowding index if ε and n are too large.

Performing, finally, an average of Eq.(1.56) in the space of trajectories, a generalization of Eq.(1.37) is obtained [42,43]:

$$\left\langle \frac{P(\vec{\varepsilon}', n)^{q-1}}{\prod_{k=1}^E \varepsilon_k^{\tau_k(q)}} \right\rangle \sim e^{-n(q-1)K(q)}, \quad (1.57)$$

where $P(\vec{\varepsilon}', n)$ stands for $P(\vec{\varepsilon}', X_0(n))$ and the product over the ε_k 's can be extracted from the average, in case they are chosen to be equal for all trajectories $X_0(n)$. Equation (1.57) has been used to evaluate fractal dimensions and metric entropies from experimental signals, by embedding a single scalar time series in an n -dimensional space [42,43]. If the embedding dimension n is larger than twice the attractor's dimension plus 1, the measured $D(q)$ and $K(q)$ agree with those of the original system [40,1].

The main obstacle to the extension of (1.57) to non-hyperbolic systems is the presence of correlations between the pointwise dimensions $\alpha_k(\mathbf{x}_0)$ and the local metric entropy $\kappa(\mathbf{x}_0, n)$ which prevents the factorization of the probability $P(\vec{\varepsilon}', X_0(n))$ into a space and a time part. This is also true for hyperbolic strange repellers (see Section 1.8).

Restricting the analysis to hyperbolic attractors, for which there is expansion at every point along the unstable directions and only contraction along the stable ones, consider again Figure 8. The probability $P(\vec{\varepsilon}', X_0(n))$ is given by the mass contained in the shaded region $B(\vec{\varepsilon}', \mathbf{x}_0)$, which is the intersection of the front face of the cylinder with the pre-image of order $n-1$ of the end face. The size of $B(\vec{\varepsilon}', \mathbf{x}_0)$ is equal to the size ε_k of the cylinder, along the contracting directions, and to $\varepsilon_k \cdot e^{-n\lambda_k(\mathbf{x}_0)}$, along the expanding directions (when n increases, the region $B(\vec{\varepsilon}', \mathbf{x}_0)$ shrinks exponentially along the latter ones). The mass $P(\vec{\varepsilon}', X_0(n))$ scales, therefore, as

$$P(\vec{\varepsilon}', X_0(n)) \sim \prod_{k=1}^E \varepsilon_k^{\alpha_k(\mathbf{x}_0)} \cdot \prod_{k=1}^{j^+} e^{-n\lambda_k(\mathbf{x}_0, n)\alpha_k(\mathbf{x}_0)}, \quad (1.58)$$

where j^+ denotes the number of expanding directions. Comparing Eq.(1.58) with Eq.(1.56), it is clear that the pointwise entropy $\kappa(\mathbf{x}_0, n)$ is given by

$$\kappa(\mathbf{x}_0, n) = \sum_{k=1}^{j^+} \lambda_k(\mathbf{x}_0, n)\alpha_k(\mathbf{x}_0), \quad (1.59)$$

where the products $\lambda_k(\mathbf{x}_0, n)\alpha_k(\mathbf{x}_0)$ can be interpreted as *partial pointwise entropies*. For hyperbolic attractors, the partial dimensions along the unstable directions are not fluctuating functions of the position ($\alpha_k = d_k(q) = 1$, $k = 1, \dots, j^+$, $\forall q$), so that after substituting Eq.(1.58) into Eq.(1.57), it is possible to factorize the average into space and time contributions: in fact, the only fluctuating quantities are the partial crowding indices of the stable directions and the effective Lyapunov exponents of the unstable ones, which are clearly independent from each

other. Finally, recalling Eq.(1.50), the relation between Lyapunov exponents and generalized metric entropies is found:

$$\left\langle \prod_{k=1}^{j^+} e^{-n(q-1)\lambda_k(\mathbf{x}_0, n)} \right\rangle \sim e^{-n(q-1)K(q)}. \quad (1.60)$$

Equation (1.59) is an extension of the Pesin formula [53] $K(1) = \sum_{k=1}^{j^+} \langle \lambda_k \rangle$, which is recovered from Eq.(1.60) in the limit $q \rightarrow 1$ (the Kolmogorov-Sinai entropy equals the sum of positive Lyapunov exponents). Relation (1.60) does not hold for strange repellers and non-hyperbolic attractors. In the latter case, in addition to the presence of partial dimensions in the expression for the entropy (1.59), two other difficulties arise: the attractor is not a direct product of continua and Cantor sets, near the homoclinic tangencies (which, however, have negligible weight in the averages) and stretching alternates with contractions along the invariant manifolds. Of course, expansion prevails along the unstable directions and contraction along the stable ones. As a consequence, some orbits may exit the cylinder of length n , considered above, at some time $i < n$ and, then, reenter it. In such a case, it is necessary to intersect the pre-image of order $i - 1$ of the cylinder's cross-section with the front face, and not the pre-image of order $n - 1$, as in the hyperbolic case. Clearly, the intermediate time $i = i(X_0(n), \bar{\epsilon})$ is a random function of the reference orbit $X_0(n)$ and coincides with n for those orbits in which only expansion occurs, along some directions. Therefore, Eq.(1.58) must be modified for non-hyperbolic systems. Notice that the number $j^+ = j^+(X_0(n))$ of expanding directions may also fluctuate. Consequently, it is possible to define a new global index j_q^+ as [49]

$$j_q^+ = \max\{j : K^{(j)}(q) \geq K^{(j-1)}(q)\}, \quad (1.61)$$

where

$$K^{(j)}(q) = \lim_{n \rightarrow \infty} \frac{1}{n(1-q)} \ln \left\langle \prod_{k=1}^j e^{-n(q-1)\lambda_k(\mathbf{x}_0, n)} \right\rangle.$$

In case this index does not vary with q , it coincides with the number of positive Lyapunov exponents. Dependence of j_q^+ on q reflects the presence of one or more effective Lyapunov exponents which assume both positive and negative values along the trajectory.

1.8 Local Exponents and Conservation Laws

In the previous two sections, global conservation laws for hyperbolic attractors have been derived. They are all based on a local asymptotic law like Eqs.(1.45) and (1.56), which is then averaged over the points of the attractor, chosen according to the invariant measure. The resulting expressions yield global quantities like the partial dimensions $d_k(q)$ (in an implicit way), or the metric entropy $K(q)$ (see Eqs.(1.50) and (1.57)). Connections with Lyapunov exponents were also found in the form of joint expectation values of static and dynamic contributions. These relations have then been simplified, by taking into account the special features of hyperbolic attractors, to yield Eqs.(1.52) and (1.60). In the case of hyperbolic repellers, however, the factorization assumption on such averages is not justified, due to correlations between crowding indices and local metric entropies. Moreover, for non-hyperbolic systems, the basic local relations (1.45) and (1.56) themselves must be modified and, in addition, correlations exist between crowding indices and effective Lyapunov exponents.

In this section, these problems are analyzed in terms of the local exponents $\alpha_k(\mathbf{x}, \varepsilon)$, $\kappa(\mathbf{x}_0, n)$ and $\lambda_k(\mathbf{x}_0, n)$.

The main features of strange attractors which allow simplifications of the conservation laws are the constancy of pointwise partial dimensions along the expanding directions and the lack of correlation between dimensions and Lyapunov exponents. The next degree of difficulty is represented by strange repellers generated by one-dimensional hyperbolic maps, for which the invariant set is Cantor-like.

Let us consider the one-dimensional repelling tent map (a modification of Eq.(1.43))

$$y_{n+1} = \begin{cases} y_n/p_1 & \text{if } 0 < y_n \leq p_1/(p_1 + p_2), \\ (1 - y_n)/p_2 & \text{if } p_1/(p_1 + p_2) < y_n < 1, \end{cases} \quad (1.62)$$

where $p_1 + p_2 < 1$. The segment $[0, 1]$ is mapped onto two segments I_1, I_2 of length p_1, p_2 , respectively, while the central part escapes from the unit interval. At the n -th iteration, a number 2^n of segments of width $p_1^i p_2^{n-i}$ is left (where i is the number of iterates spent in the interval I_1). Asymptotically, the only non-escaping points belong to a non-uniform Cantor set (see Section 1.3) which can also be obtained by backward iteration of the map (1.62) (every point has two pre-images). At variance with strange attractors, here there is a certain arbitrariness in the choice of the invariant measure. The *natural* measure $\rho(y)$ is obtained by weighting the pre-images proportionally to their own width (p_1 or p_2), at each step. More precisely, $\rho(y)$ is defined as the fixed-point solution of the modified Frobenius-Perron equation [54,55]

$$\rho_{n+1}(y') = e^\beta \sum_{y \in F^{-1}(y')} \frac{\rho_n(y)}{|F'(y)|} \quad (1.63)$$

where $F'(y)$ is the derivative of the map. A finite limit $\rho(y) > 0$ only exists for $\beta = -\ln(p_1 + p_2) > 0$. The measure $\rho(y)$ can be seen as the limit, for $n \rightarrow \infty$, of

piecewise constant densities [56]

$$\rho_n(y) = \begin{cases} e^{n\beta} & \text{if } F^n(y) \in [0, 1] , \\ 0 & \text{otherwise.} \end{cases}$$

The exponent β is called escape rate and is related to the fraction f^n of mass which has not yet escaped a neighbourhood of the repeller after n steps: $f^n = e^{-n\beta}$. In Eq.(1.63), the term e^β is a global compensation factor for the outflow of probability from the repeller ($\beta = 0$, for attractors). Repellers can be considered as models of transient behaviour in dynamical systems, the escape corresponding to the landing of the orbit on some stable solution.

The main difference with strange attractors is that, now, the unstable directions present a fractal structure: for the map (1.62), the (global) dimension $d_1(q)$ is given by the self-similarity relation (1.17), with $M = 2$ and the proper rates and weights. In order to see the connection between the local exponents (crowding indices and effective Lyapunov exponents), let us consider the segment $[0, 1]$, at time n , containing a unit mass. One backward iteration of the map yields two intervals of lengths p_1 and p_2 and masses $p_1 e^\beta$ and $p_2 e^\beta$ (the natural measure is chosen): in fact, the mass decreases by a factor $e^{-\beta}$ at each forward iteration. After n backward iterations, a generic segment, centered on y_0 , will have size

$$\varepsilon(n) = e^{-n\lambda_1(y_0, n)} \quad (1.64)$$

and mass $P(\varepsilon(n), y_0) = p_1^i p_2^{n-i} e^{n\beta} = e^{-n[\lambda_1(y_0, n) - \beta]}$. The escape rate β is, then, related to the crowding index $\alpha_1(y_0, \varepsilon) = \ln P(\varepsilon, y_0) / \ln \varepsilon$ by

$$\beta = [1 - \alpha_1(y_0)] \lambda_1(y_0, n) , \quad (1.65)$$

and the local metric entropy $\kappa(y_0, n)$ is

$$\kappa(y_0, n) = -\frac{\ln P(\varepsilon(n), y_0)}{n} = \lambda_1(y_0, n) - \beta = \alpha_1(y_0) \lambda_1(y_0, n) . \quad (1.66)$$

For hyperbolic strange attractors, $\alpha_1 = 1$ everywhere and $\beta = 0$. Notice that, in the model (1.62), β is not a fluctuating variable, differently from α_1 and λ_1 . Moreover, the first partial crowding index $\alpha_1(y_0)$ depends on the value of the Lyapunov exponent computed over the trajectory starting from y_0 (i.e., it depends on the future evolution). It is therefore clear that conservation laws like Eq.(1.51) cannot factorize, in general, if α_1 is not constant, since they also involve effective Lyapunov exponents computed over the future n iterations of the reference point.

The local approach also allows to derive relations for the partial dimensions of a generic system. Considering an ellipsoid $B(\bar{\varepsilon}, \mathbf{x})$, centered at $\mathbf{x}$, its image $B(\bar{\varepsilon}', \mathbf{x}')$ (with $\mathbf{x}' = \mathbf{F}^n(\mathbf{x})$) will be an ellipsoid with axes ε'_k given by Eq.(1.39), containing the same mass: $P(\bar{\varepsilon}, \mathbf{x}) = P(\bar{\varepsilon}', \mathbf{x}')$. For periodic orbits, since $\alpha(\mathbf{x}) = \alpha(\mathbf{x}')$, the latter relation, together with Eqs.(1.39) and (1.45), yields [30]

$$\sum_{k=1}^E \alpha_k(\mathbf{x}) \lambda_k(\mathbf{x}, n) = 0 \text{ (periodic orbits)} . \quad (1.67)$$

This relation is the local version of Eq.(1.54). For non-periodic trajectories, it holds nearly everywhere (since $\lambda_k(\mathbf{x}) = \langle \lambda_k \rangle$ at nearly all points $\mathbf{x}$). However, an extension of Eq.(1.54), valid for any $\mathbf{x}$, is only possible for natural measures on hyperbolic attractors, for which no alternation of expansion and contraction occur, along any direction. Then, by choosing $\varepsilon_k = O(1)$, for the stable directions, and $\varepsilon'_k = O(1)$, for the unstable ones, it is possible to proceed as in the derivation of Eq.(1.67). Taking the limit $n \rightarrow \infty$, with the end point $\mathbf{x}'$ kept fixed, we obtain [30]

$$\sum_{unstable} \lambda_k(\mathbf{x}', -\infty) = - \sum_{stable} \alpha_k(\mathbf{x}') \lambda_k(\mathbf{x}', -\infty), \quad (1.68)$$

where $\lambda_k(\mathbf{x}', -\infty)$ is the Lyapunov exponent of the trajectory leading to $\mathbf{x}'$ and the constancy of the partial dimensions along the unstable directions has been taken into account ($\alpha_k(\mathbf{x}) = 1$, $k = 1, \dots, j^+$). For the baker map (1.44), one has

$$\alpha_2(\mathbf{x}_n) = -\lambda_1(\mathbf{x}_0, n) / \lambda_2(\mathbf{x}_0, n), \quad (1.69)$$

for any n .

The partial pointwise dimensions along the contracting directions depend on the Lyapunov exponents computed over the past history of the reference point, at variance with those along the expanding directions. This justifies the assumption on factorization after Eq.(1.51), as already anticipated in Section 1.6.

Part II

Thermodynamic Formalism

In the first part, we have derived conservation laws for dissipative dynamical systems as suitable averages of “local” relations involving pointwise dimensions, metric entropies and effective Lyapunov exponents.

The global approach, consisting of the evaluation of such averages, provides a “macroscopic” description of the system in terms of generalized dimensions $d_k(q)$ (Eqs.(1.49) and (1.50)) and metric entropies $K(q)$ (Eq.(1.57)), yielding relations between them and Lyapunov exponents (Eqs.(1.51) and (1.60)). The local approach, discussed in Section 1.8, constitutes the “microscopic” part of the theory. These two aspects are formally equivalent to thermodynamics and statistical mechanics, respectively. The former theory, in fact, describes macroscopic systems through a set of intensive variables ($1/k_B T, P/k_B T, -\mu_k/k_B T$) and their conjugate extensive parameters (E, V, N_k) [57] while, in statistical mechanics, the thermodynamic functions are derived from the knowledge of the hamiltonian.

The formal analogy between one-dimensional statistical mechanics and dynamical systems was originally developed by Sinai, Ruelle and Bowen [58] for one-dimensional expanding maps and Julia sets (see also [59]). This approach is here extended to generic (i.e., also non-hyperbolic) multi-dimensional systems displaying chaotic solutions. The most interesting aspect of this analysis is given by the existence of non-analytic behaviour of the dimension function $D(q)$, which can be interpreted as an evidence of a phase-transition-like behaviour on the time lattice. Such singularities have been so far observed in some low-dimensional dynamical systems of physical relevance: the circle map at the onset of chaos [60], the logistic equation [61,30] and the Hénon map [30]. In this Part, we introduce various equivalent sets of definitions of “thermodynamic” quantities, stressing the connections with different covering procedures of fractal measures. In particular, the choice of either fixed-size or fixed-mass coverings is shown to correspond to a change of sign in the “temperature” (and, therefore, to the passage from ferromagnetic to antiferromagnetic interactions).

In Section 2.1, the “microscopic” description of nonlinear dynamical systems is presented; in Section 2.2, the “thermodynamic” functions are derived from generalized statistical ensembles (Microcanonical, Canonical, Isothermal-Isobaric Ensemble, etc.), and related to the *dimension spectrum* $f(\alpha)$.

In Section 2.3, the conditions under which a phase-transition can occur for a spin system on a one-dimensional lattice are reviewed. The implications for the associated dynamical system are studied through the behaviour of the dimension function $D(q)$ and of the dimension spectrum $f(\alpha)$. Finally, in Section 2.4, a similar analysis is performed for metric entropies and Lyapunov exponents.

2.1 Microscopic Description

Let us assume that a generating partition composed of M elements has been detected, so that each point x in phase-space can be univocally represented by an infinite sequence of symbols $\{\dots, s_{i-1}, s_i, s_{i+1}, \dots\}$, with $s_i \in [0, M-1]$. For sim-

licity, consider a two-dimensional map with a binary partition ($M = 2$). It is, then, necessary to choose a covering and assign a symbolic sequence to every element of it. In particular, we take ellipsoids $B(\vec{\varepsilon}, \mathbf{y})$ of axes $\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2)$, centered at $\mathbf{y}$. The size along the contracting direction must be small enough to guarantee that the first n_2 pre-images of all the points in the ellipsoid generate the same backward symbolic sequence S_{n_2} : for the baker map, this yields $\varepsilon_2 \approx e^{n_2 \lambda_2(\mathbf{y}, -n_2)}$, where $\lambda_2(\mathbf{y}, -n_2)$ is the second effective Lyapunov exponent computed over the first n_2 pre-images of $\mathbf{y}$. Analogously, along the expanding directions, the size will be such that all points yield the same forward symbol sequence T_{n_1} , of length n_1 : i.e., $\varepsilon_1 \approx e^{-n_1 \lambda_1(\mathbf{y}, n_1)}$. In the following, S_n (respectively, T_m) will denote a backward (forward) sequence of length n (m). This procedure leads, in general, to a covering in which both the mass P and the size ε' of the balls fluctuate with the position. However, in the next sections, coverings will be used in which either P or ε' will be fixed, for all reference points $\mathbf{y}$. The numbers n_1 and n_2 will still be determined as discussed above (largest number of forward or backward iterates for which the symbolic sequence remains the same for all points in the ball): in fact, as seen in Section 1.8, the partial dimensions along the contracting directions depend on the past history of the reference point, while the contrary is true for the expanding directions.

In this way, a finite sequence $S_{n_2} \cup \{s_0\} \cup T_{n_1} = \{s_{-n_2}, \dots, s_0, \dots, s_{n_1}\}$ labels the ball $B(\varepsilon, \mathbf{y})$ (the symbol s_0 corresponds to the point $\mathbf{y}$ itself).

In the mapping onto a thermodynamic formalism [58], time is considered as a one-dimensional lattice with sites labelled by the index $i \in \{\dots, -2, -1, 0, 1, 2, \dots\}$. Accordingly, a sequence S_n is interpreted as a chain of n spins σ_i , which are defined as $\sigma_i = s_i - (M - 1)/2 \in [-(M - 1)/2, (M - 1)/2]$. Therefore, each ball of the covering of the set is associated, through its symbolic sequence, to a one-dimensional configuration of spins and represents the *microstate*.

For the evaluation of the dimension function it is necessary to perform a suitable average (see Eq.(1.22)) over a set of reference points $\mathbf{y}$, sampled from the fractal at random with respect to the natural measure. Restricting the discussion, for simplicity, to hyperbolic two-dimensional transformations, where the expanding direction is smooth (non-fractal), it is possible to describe the Cantor direction through

$$\left\langle \frac{P^{q-1}}{\varepsilon^\tau} \right\rangle = \int d\rho(\mathbf{y}) P(\varepsilon, \mathbf{y})^{q-1} / \varepsilon(\mathbf{y})^\tau, \quad (2.1)$$

where $\varepsilon = \varepsilon_2$. Notice that an extra weight P appears in the case of box-counting (Eqs.(1.15) and (1.18)), where the reference points are chosen independently of their occurrence probability. In Eq.(2.1), instead, all microstates are equiprobable (see, e.g., [57] for a discussion on the principle of the equiprobability a priori): simply, there is a larger number of reference points $\mathbf{y}$ in a region in which the probability is higher (the same discussion applies to Eqs.(1.18) and (1.22)). Each representative point $\mathbf{y}$ contributes to the sum with a term $P^{q-1}(S_n)/\varepsilon^\tau(S_n)$, which is a function of the parameters q and τ and of the configuration S_n (given by the symbols corresponding to the n pre-images of $\mathbf{y}$). Similar considerations hold for

the other conservation laws (Eqs.(1.50), (1.51) and (1.57)).

In the next section, it will be shown how statistical ensembles can be constructed by interpreting the averages as partition functions. The extensive thermodynamical variables ($E, V, \dots$) will then be defined in terms of the probabilities $P(S_n)$ and of the radii $\varepsilon(S_n)$. To this end, in Refs. [27,61,62,63] a fixed-mass covering was adopted and the parameter $-\tau$ was identified with the inverse of the temperature $\beta \equiv 1/k_B T$ (where k_B is the Boltzmann constant). However, that choice leads to define the pressure as a negative quantity. Moreover, due to the “symmetry” of Eq.(2.1) in the parameters τ and q^{-1} , the choice of the temperature (as well as that of the energy E and of the volume V) is completely arbitrary and is a matter of convenience. In the next section, we shall describe both conventions and relate generalized statistical ensembles to the kind of covering adopted.

To show, in a simple case, how dynamical quantities can be expressed through “spin”-values, let us consider the crowding index $\alpha(S_n)$ of a subinterval of the non-uniform Cantor set (see Section 1.3) specified by a symbol-sequence $S_n = \{s_{-n}, \dots, s_{-1}\}$. The crowding index $\alpha = \ln P / \ln \varepsilon$ is given by

$$\alpha(S_n) = \frac{n \ln \sqrt{p_1 p_2} + \sum_i \sigma_{-i} \ln(p_2/p_1)}{n \ln \sqrt{r_1 r_2} + \sum_i \sigma_{-i} \ln(r_2/r_1)} \quad (2.2)$$

and depends on the sum $\sum_i \sigma_{-i}$ of the spin values in the sequence ($\sigma_{-i} = \pm 1/2$). In the “thermodynamic” limit $n \rightarrow \infty$, the sum tends to $n(p_2 - p_1)/2$ and Eq.(2.2) reduces to $\alpha = D(1) = (p_1 \ln p_1 + p_2 \ln p_2)/(p_1 \ln r_1 + p_2 \ln r_2)$: i.e., the pointwise dimension is nearly everywhere equal to the information dimension $D(1)$. In the next section, we shall study the asymptotic shape of the distribution of crowding indices, in the limit $n \rightarrow \infty$.

2.2 Statistical Ensembles

In this section, the evaluation of the dimension function $D(q)$ for the Cantor direction of a two-dimensional hyperbolic map will be considered, while the extension to higher-dimensional systems will be done in Section 2.4. Let us perform a covering with fixed-size balls, centered on points of the set, chosen at random from the natural probability measure $\rho(x)$. The dimension function $D(q)$ is, then, given by Eq.(1.23), which can be rewritten as follows:

$$Z_{can}(V, T) \equiv e^{-F(V, T)/k_B T} = \sum_{\{s_{-n}, \dots, s_{-1}\}} e^{-(1-q)\alpha(s_{-n}, \dots, s_{-1}) \ln \epsilon} \sim e^{-(1-q)D(q) \ln \epsilon}, \quad (2.3)$$

where Z_{can} is the canonical partition function, F the free energy, V the volume and T the temperature. The sum is over all sequences (i.e., balls) S_n : the only variable which depends on the position is the probability $P(\epsilon) \sim \epsilon^\alpha$, through the crowding index $\alpha = \alpha(S_n)$. The partition function (2.3) is unnormalized (as all the others to be introduced in the following). By defining energy, temperature and volume as

$$\begin{aligned} E &= -\ln P(S_n) = \alpha(S_n) \ln \epsilon < 0, \\ \beta &= 1/k_B T = 1 - q, \\ V &= -\ln \epsilon > 0, \end{aligned} \quad (2.4)$$

we obtain

$$F(V, T) = D(q) \ln \epsilon = -VD(q) \equiv V\mathcal{F}(T), \quad (2.5)$$

where $\mathcal{F}(T) = -D(q)$ is the free energy per unit volume. Therefore, in the canonical description, the fractal is seen as a collection of balls of fixed size (each one characterized by its crowding index α) representing an ensemble of spin systems with energies $E = \ln P(S_n)$. Of course, many microscopic states $S_n = \{s_{-n}, \dots, s_{-1}\}$ possess the same energy (all the permutations of a given sequence yield the same probability). The macroscopic equilibrium properties of the system are given by the averages of phase functions (i.e., functions of the configuration S_n) over the virtual ensemble specified by the partition function (2.3) [57].

The microcanonical description is obtained by considering the ensemble consisting of all sequences which yield an energy $E = \ln P(S_n)$ between E and $E + dE$ or, since ϵ is fixed, a dimension $\alpha(S_n) \in [\alpha, \alpha + d\alpha]$. The number of times we find $\alpha(S_n) \in [\alpha, \alpha + d\alpha]$ in the evaluation of the sum (2.3) (equivalent to the average (1.23)) is proportional to the probability $P(\alpha; \epsilon) d\alpha$ to find $\alpha(S_n) \in [\alpha, \alpha + d\alpha]$ [30,49], since the reference points y are chosen at random from the natural measure. Hence, we can define the microcanonical partition function $Z_{mic}(E, V)$ as

$$Z_{mic}(E, V) \equiv e^{S(E, V)/k_B} \propto P(\alpha; \epsilon) \sim \sqrt{|\ln \epsilon|} \epsilon^{\alpha - f(\alpha)}, \quad (2.6)$$

where $S(E, V)$ is the microcanonical entropy. The function $f(\alpha)$, introduced in [64], and here defined through relation (2.6), is called spectrum of dimensions

and can be interpreted as the Hausdorff dimension of the set of points $\mathbf{y}$ which display the same pointwise dimension α [27,64,65,30]. In fact, in the case of box-counting, where points are sampled independently of their occurrence probability, the number of boxes with pointwise dimension α scales as $\varepsilon^{-f(\alpha)}$ [27]. The presence of the factor ε^α in Eq.(2.6) is due to the sampling adopted in Eqs.(2.1) and (2.3) and balances the extra power of P needed when box-counting is performed (Eqs.(1.15) and (1.18)). The microcanonical entropy is, then, given by

$$S(E, V) = -k_B [f(\alpha) - \alpha] \ln \varepsilon = VS(\mathcal{E}), \quad (2.7)$$

where the prefactor $\sqrt{|\ln \varepsilon|}$ in the expression of $P(\alpha; \varepsilon)$ has been neglected, in the limit $\varepsilon \rightarrow 0$. The quantity $\mathcal{E} = E/V = -\alpha$ (see Eq.(2.4)) is the specific energy and $S(\mathcal{E}) = k_B [f(\alpha) - \alpha] = k_B [f(-\mathcal{E}) + \mathcal{E}]$ is the specific entropy. By using Eq.(2.6), the sum (2.3) can be evaluated as an integral:

$$Z_{can}(V, T) = \sqrt{|\ln \varepsilon|} \int e^{[\alpha - f(\alpha)] \ln \varepsilon} e^{(q-1)\alpha \ln \varepsilon} d\alpha \sim e^{\tau(q) \ln \varepsilon}. \quad (2.8)$$

In the limit $\varepsilon \rightarrow 0$, the integral is dominated by the value $\alpha(q)$ of α which yields the minimum of the exponent $q\alpha - f(\alpha)$. Therefore, we obtain

$$\tau(q) = \inf_{\alpha} \{q\alpha - f(\alpha)\}, \quad f(\alpha) = \inf_q \{q\alpha - \tau(q)\}, \quad (2.9)$$

i.e. the function $f(\alpha)$ is the Legendre transform of $\tau(q)$. In particular, if $f(\alpha)$ and $\tau(q)$ are differentiable, one gets the relations

$$f(\alpha) = q\alpha(q) - \tau(q), \quad \alpha(q) = \frac{d\tau(q)}{dq}, \quad q = \frac{df(\alpha)}{d\alpha}. \quad (2.10)$$

If the minima in Eq.(2.9) are attained at a point in which $f(\alpha)$ is not differentiable, but has finite left and right derivatives, the prefactor $\sqrt{|\ln \varepsilon|}$ has to be replaced with $|\ln \varepsilon|$ [30].

The first of Eqs.(2.10) can be rewritten as $D(q) = \alpha(q) - [f(\alpha) - \alpha(q)]/(q-1)$, which is the analogue of the thermodynamic relation

$$\mathcal{F}(T) = \mathcal{E}(T) - T S(\mathcal{E}), \quad (2.11)$$

as follows from Eqs.(2.4), (2.5) and (2.7). In the following, the differential form of the Legendre transform (Eqs.(2.10) and (2.11)) will be used, replaced by the more general relation (2.9) when necessary (see, e.g., [66]). All usual thermodynamic relations hold: for example, from Eq.(2.7), we have

$$\left(\frac{\partial S}{\partial E} \right)_V = \frac{1}{T}, \quad \left(\frac{\partial S}{\partial V} \right)_E = \frac{P}{T}, \quad \left(\frac{\partial E}{\partial S} \right)_V = T \quad \text{and} \quad \left(\frac{\partial E}{\partial V} \right)_S = -P, \quad (2.12)$$

where $P = D(q)$ is the pressure. Moreover, it is easy to verify that S is a concave function of E and V , E is a convex function of S and V , and so on [66].

The function $f(\alpha)$ is concave in α , and its maximum $\max_{\alpha}\{f(\alpha)\}$ is the Hausdorff dimension of the (distribution-theoretic) support of ρ . The minimal and maximal values of α for which $f(\alpha) \neq -\infty$ are indicated with $\alpha_{\min}$ and $\alpha_{\max}$ and coincide with the smallest and the largest pointwise dimensions of the set, respectively. Finally, $f(\alpha)$ is tangent to the curve $f(\alpha) = \alpha$, at $\alpha = D(1)$ (see Fig. 9). Other generalized ensembles can be obtained by successively relaxing certain con-

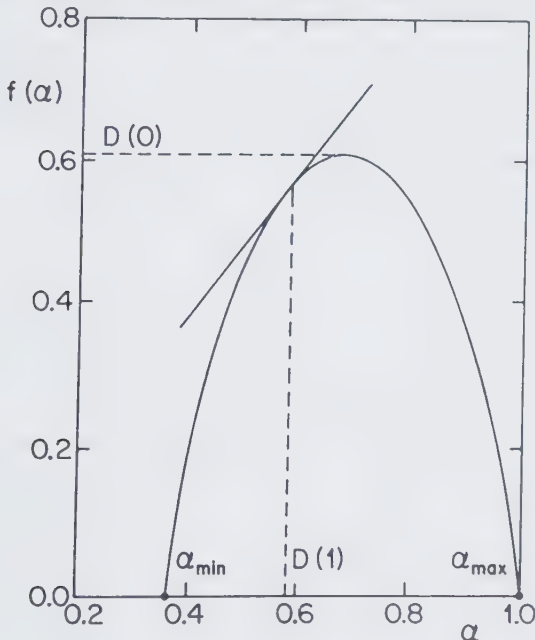


Figure 9: Dimension-spectrum $f(\alpha)$ vs. α , for the non-uniform Cantor set with $r_1 = 0.25$, $r_2 = 0.4$, $p_1 = 0.6$ and $p_2 = 0.4$.

ditions. For example, by performing a covering where both masses P and ball-sizes ϵ are allowed to fluctuate with the position $S_n = \{s_{-n}, \dots, s_{-1}\}$, we can construct the isothermal-isobaric partition function [57,66]

$$Z_{iso}(P, T) \equiv e^{-G(P, T)/k_B T} = \sum_{\epsilon(S_n)} \sum_{\alpha(S_n)} e^{-(1-q)\alpha(S_n) \ln \epsilon(S_n)} e^{-\tau(q) \ln \epsilon(S_n)}, \quad (2.13)$$

where $G(P, T)$ is the Gibbs free energy and $-\tau(q) = P/k_B T$, in agreement with Eq.(2.12). This ensemble is also called “pressure ensemble” [57]. Notice that the fluctuations of the “volume” $-\ln \epsilon(S_n)$ and of the “energy” $\ln P(S_n)$ are not

completely independent, since the variable $\varepsilon(S_n)$ appears in both the definitions. However, after the evaluation of the sum over all ε 's, residual fluctuations of $E = \ln P(S_n)$ persist, due to presence of the crowding index $\alpha(S_n)$ (over which we still have to sum). Notice that the "pressure" $D(q)$ is positive since the energy has been defined as a negative quantity in Eq.(2.4), at variance with Refs.[61,62,63]. An additional difference with [61,62,63] is in the choice of the reference balls (at random with respect to the natural measure ρ and not to Lebesgue's measure). This leads to defining the inverse temperature β as $1 - q$ and not as $-q$. A consequence of this, as already discussed, is that the entropy $S(E, V)$ is proportional to $f(\alpha) - \alpha$ and not to $f(\alpha)$. The function $f(\alpha) - \alpha \leq 0$ is concave and has its maximum (equal to 0) at $\alpha = D(1)$. In terms of the probability density (2.6), this means that the "most probable" pointwise dimension coincides with the information dimension $D(1)$, as discussed in the previous section.

In the case of the non-uniform Cantor set (see Eq.(1.44)), the "energy" is given by $E = n \ln \sqrt{p_1 p_2} + \sum_i \sigma_{-i} \ln(p_2/p_1)$ (with $\sigma_{-i} = \pm 1/2$) and is equivalent to the expression of the hamiltonian of an Ising model without exchange interaction, but in a magnetic field. The absence of spin-spin interactions is a consequence of the lack of correlation between successive symbols, in the baker map (see Section 1.6). The expression for the "volume" V is analogous to that for the energy. The non-uniform Cantor set can also be studied in the framework of the canonical formalism, by choosing a constant-size covering: if ε is fixed, the whole dependence on the spin values is transferred to the energy E which, thus, will also depend on the parameters r_1 and r_2 , although in a complicated way.

If the fixed-mass covering (1.24) is employed, with the convention (2.4), still another ensemble is obtained for which the Massieu-Planck function is $S - PV/T$ [57]. If other constraints are progressively relaxed allowing, for example, the number of iterations n (or the escape rate of strange repellers [55,67]) to fluctuate, new generalized ensembles are constructed. The conservation laws can be expressed through properties of the partition functions, which are obtained one from the other by means of Laplace transforms, while the corresponding Massieu-Planck functions are related through Legendre transforms [57]. Other examples will be presented in Section 2.4.

In many cases, it is convenient to work with constant-mass coverings since the analytic expression of the ball sizes ε can be computed, either exactly or asymptotically. Therefore, to facilitate the interpretation of the results in terms of hamiltonian spin-systems, it is useful to adopt a different definition of energy, temperature and volume than in Eq.(2.4). In this approach (which coincides with that of Refs.[61,62,63], apart from the sign of E and T), the set is covered with balls of fixed mass p and the canonical partition function is

$$Z_{can}(V, T) = \sum_{\{S_n\}} e^{-(\tau/\alpha(S_n)) \ln p} \sim e^{[1-q(\tau)] \ln p}, \quad (2.14)$$

where the same symbols as in Eq.(2.3) and following are used for all "thermody-

namic” quantities, although they are, now, defined differently:

$$\begin{aligned} E &= \ln \varepsilon(S_n) < 0, \\ \beta &\equiv 1/k_B T = \tau, \\ V &= -\ln p \geq 0. \end{aligned} \quad (2.15)$$

The free energy is $F(V, T) = \ln p/D(q) = -V/D(q)$ and $\mathcal{F}(T) = -1/D(q)$. Notice that $q(\tau)$, in Eq.(2.14), is the inverse function of $\tau(q)$ and that F depends on the temperature $1/\tau$ through $q = q(\tau)$.

The microcanonical ensemble consists of all balls with size $\varepsilon(S_n) = p^{1/\alpha(S_n)}$ in the range $[\varepsilon, \varepsilon + d\varepsilon]$. The microcanonical partition function is, then, defined in terms of the probability density $\hat{P}(\alpha; p)$ to observe a crowding index α in a ball of mass p , chosen at random with respect to $\rho(\mathbf{x})$ [30]:

$$Z_{mic}(E, V) \equiv e^{S(E, V)/k_B} \propto \hat{P}(\alpha; p) \sim \sqrt{|\ln p|} p^{1-f(\alpha)/\alpha}, \quad (2.16)$$

in analogy with Eq.(2.6). Taking the limit $p \rightarrow 0$, we obtain the entropy

$$S(E, V) = -k_B [f(\alpha)/\alpha - 1] \ln p = VS(\mathcal{E}), \quad (2.17)$$

with $\mathcal{E} = E/V = -1/\alpha$ and $S(\mathcal{E}) = -k_B [\mathcal{E} f(-1/\mathcal{E}) + 1]$. Relations (2.9)-(2.11) are still valid, as well as Eq.(2.12), with $P = 1/D(q)$.

The main difference with the previous set of definitions is the sign of the temperature: in Eq.(2.15), $\beta = \tau(q) = (q - 1)D(q)$ while, in Eq.(2.4), $\beta = 1 - q$. In both cases the energy is negative. Notice that, in Refs.[61,62,63], β was defined as $-\tau$ and the energy as a positive quantity (as already mentioned, this choice would, however, yield a negative pressure P).

All partition functions have so far been defined in such a way that the correct asymptotic relations (2.3), (2.6), (2.14), (2.16) hold, in the limit of infinitely long symbol-sequences. To extract the large- n behaviour of the energy E or of the volume V and minimize possible “transient” effects (i.e., small- n deviations), it is sometimes useful to consider two successive orders of approximation of the fractal, represented by the sequences S_n and S_{n+1} . In the case of fixed-mass coverings (2.14), it has been proposed [68,61,63] to consider the *scaling function* [69]

$$\hat{\sigma}(s_{-n}, \dots, s_0) = \varepsilon(s_{-n}, \dots, s_0)/\varepsilon(s_{-n}, \dots, s_{-1}) \quad (2.18)$$

and to define the energy as $E = \ln \hat{\sigma}(s_{-n}, \dots, s_0)$. For sets which are not self-similar, the scaling function (which coincides with the multiplier at time $n = 0$) depends on the whole sequence S_{n+1} : for the baker map (exactly self-similar), it only depends on the last symbol s_0 , which corresponds to the reference point y itself. The free energy is, then, determined from the largest eigenvalue of the transfer matrix [68,61,63]

$$\langle s_{-n+1}, \dots, s_0 \mid T \mid s_{-n}, s'_{-n+1}, \dots, s'_{-1} \rangle = \hat{\sigma}^{-\tau}(s_{-n}, \dots, s_0) \delta_{s_{-n+1} s'_{-n+1}} \cdot \dots \cdot \delta_{s_{-1} s'_{-1}}. \quad (2.19)$$

The scaling function $\hat{\sigma}(s_{-n}, \dots, s_0)$ may depend more strongly on the low-order symbols $(s_{-n}, s_{-n+1}, \dots)$ or on the high-order ones $(\dots, s_{-1}, s_0)$, according to the problem and to the covering [63]. The range of the interactions in the hamiltonian $H = \ln \hat{\sigma}$ is determined by the correlations between the spin values at different lattice sites, $\langle s_n s_{n+m} \rangle$, which affect the probability weights $P(S_n)$ and the ball sizes $\varepsilon(S_n)$. In the next section, we shall discuss the conditions which allow a non-analytic behaviour of the thermodynamic functions $F(V, T)$ or $P(T)$, as a function of T , for one-dimensional lattice systems.

2.3 Phase Transitions on One-dimensional Lattices

The interpretation of the conservation laws in terms of a thermodynamic formalism allows the use of a unified language for the description of strange attractors and other statistical mechanical systems like Ising spin-configurations, turbulence, percolating clusters, diffusion-limited aggregates or, more in general, of critical phenomena [70]. The duality between the functions $D(q)$ and $f(\alpha)$, expressed by the Legendre transform (2.9), indicates that a fractal measure can be either characterized by specifying the moments of the distribution of the random variable $P(\epsilon, \mathbf{x})$ (in the case of fixed-size coverings, e.g.) or by evaluating the dimension spectrum $f(\alpha)$, from the definition (2.6). The two procedures are completely

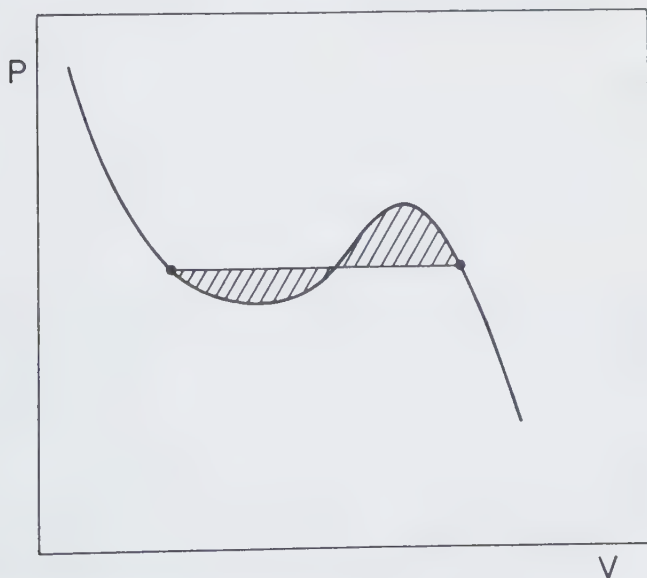


Figure 10: Isothermal curve of a liquid-vapor system. The shaded regions have equal areas (Maxwell's construction). The extrema of the segment correspond to pure states.

equivalent, although their meaning is different: the former one corresponds to the characterization of the system by means of its free energy while, in the latter case, the entropy is given, as a function of the energy E and volume V .

The construction of this formal apparatus finds its major justification in the

application to those dynamical systems for which a non-analytic behaviour of the dimension function $D(q)$ may occur, at some parameter values. This phenomenon can be interpreted as a phase-transition since it is equivalent to a discontinuity in some derivative of the free energy (2.5) (or of the pressure) [57,66]. In fact, the equilibrium states of a thermodynamical system are represented by the graph of the function $E(S, V)$ (or of $S(E, V)$) and the coexistence of different phases corresponds to flat portions of the surface $E(S, V)$: the s-shaped region of the isothermal curve $P(V)$ is replaced by a horizontal straight line, in such a way that the areas above and below it are equal (Maxwell's equal area rule) [66]. In figure 10, a schematic picture of an isothermal of a liquid-vapor system is presented: the extrema of the segment correspond to pure phases and the internal points to mixtures of liquid and gas phases. A discontinuous change in the derivative occurs at the boundary. The implications of this for fractal measures can be discussed by referring to rela-

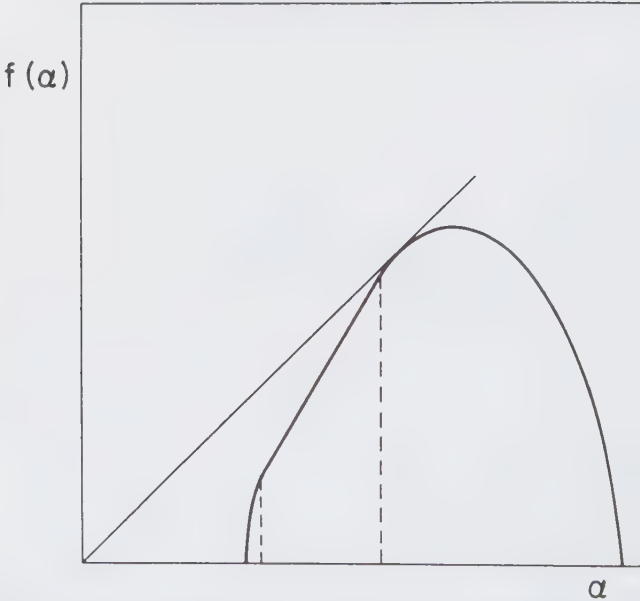


Figure 11: Schematic plot of the function $f(\alpha)$ vs. α , for a system exhibiting a "phase-transition". All α 's between the two dashed lines represent "mixed-states" coexisting at the same "temperature" $q = df(\alpha)/d\alpha$.

tion (2.9): a corner-point in the "free energy" $D(q)$ is Legendre-transformed to a segment in the graph of the "entropy" $f(\alpha)$ (more precisely $f(\alpha) - \alpha$). Since the "temperature" $1/(1 - q)$ is essentially given by the slope of the curve $f(\alpha)$ vs. α

($q = f'(\alpha)$, Eq.(2.10)), all sequences S_n corresponding to a pointwise dimension located below that segment represent mixed states, coexisting at the same temperature, but possessing different "entropies" $f(\alpha) - \alpha$. This situation is illustrated in Fig. 11. Notice that whenever there is a flat part in a thermodynamic function, the full Legendre transform (2.9) must be used: relation (2.10) only holds for strictly convex energy (or strictly concave entropy) [66].

Let us now examine the conditions under which dynamical systems can give rise to this phenomenology: in fact, they can be mapped onto one-dimensional spin-models which are known not to exhibit phase-transitions, if the interactions are of the short-range type [71]. To be rigorous, it is not even correct to speak about separate phases in one-dimensional substances, for any choice of the potential [71,72]. Notwithstanding this, interesting phenomena take place, for certain choices of long-range interactions (Tonk's gas plus a Van der Waals attraction [71], Fisher's droplet model [73]). In particular, for a one-dimensional Ising ferromagnet with hamiltonian

$$H = - \sum_{i>j} J(i-j) \sigma_i \sigma_j, \text{ with } J(n) \geq 0, \quad (2.20)$$

it has been proven that a phase-transition (i.e., long-range order) exists if the interaction $J(n)$ decays sufficiently slowly with n , including the case $J(n) \sim n^{-\nu}$ with $1 < \nu < 2$ [74]. Further refinements of this proof have been found [75]. Moreover, a class of lattice gas models with many-body interactions has been shown to present a discontinuous behaviour of the pressure, as a function of the density (at fixed temperature), under certain restrictions on the inter-particle potential [76].

Translated into the language of dynamical systems, such a behaviour is possible either if there are long-time correlations between symbols or if the generating partition has infinitely many elements (i.e., if the spins have infinitely many states: $M = \infty$). Of course, memory effects in the scaling function $\hat{\sigma}(S_n)$ are only possible for sets which are not self-similar. In such a case, as remarked in Section 1.3, the partition function at the n -th iteration cannot be simply written as the n -th power of a known generator (see Eq.(1.20)). So far, phase-transition-like behaviour has been detected in simple one-dimensional non-hyperbolic transformations (circle map at the onset of chaos [60], logistic equation [61,30]) and in the Hénon map [30] (two-dimensional, non-hyperbolic), thus suggesting that the main feature responsible for the long-range order may be the non-hyperbolicity. In Part III, in addition to the above mentioned examples, we shall discuss an exactly soluble three-dimensional hyperbolic system displaying two different kinds of transition. This system is obtained by filtering the baker map (1.44) and presents solutions which change structure (from self-similar to self-affine, for instance) upon variation of a parameter [77,49]. In that case, the transitions have to be attributed to the occurrence of tangencies between the eigendirections of the linearized map in the contracting subspace. From this example, it will follow that non-hyperbolicity is not necessary to produce discontinuities in the dimension function $D(q)$.

2.4 Spectra of Metric Entropies and Lyapunov Exponents

In addition to the dimension spectrum $f(\alpha)$, it is possible to define a spectrum $g(\kappa)$ for the metric entropy κ and a spectrum $h(\bar{\lambda})$ for the effective Lyapunov exponents $\bar{\lambda}$, to characterize the scaling properties of the probability distributions of these quantities in the infinite-time limit. Let us first consider, for simplicity, relation (1.60) rewritten for a one-dimensional hyperbolic map:

$$Z_{\text{can}}(\mathbf{V}, T) \equiv \left\langle e^{-n(q-1)\kappa(\mathbf{x}, n)} \right\rangle \sim e^{-n(q-1)K(q)}, \quad (2.21)$$

where $\kappa(\mathbf{x}, n)$ is the local entropy of the portion of orbit consisting of the first n iterates of the point $\mathbf{x}$ (symbolic sequence T_n) and coincides with the effective Lyapunov exponent $\lambda(\mathbf{x}, n)$. The average (2.21) can evidently be interpreted as a canonical partition function where the energy is given by $E = -n\kappa(T_n) < 0$. The volume is, now, $V = n$ and the inverse temperature is $\beta = 1 - q$; in fact, a constant-size covering in trajectory-space is adopted, here, since the length n of the orbit is fixed. The free energy is, then, $F(V, T) = -nK(q)$. Notice that the energy is again defined as a negative quantity (see Section 2.2). The average (2.21) can be evaluated as an integral by introducing the probability density $P(\kappa; n)$ to find a pointwise entropy $\kappa(n) \in [\kappa, \kappa + d\kappa]$ and by defining the microcanonical partition function as

$$Z_{\text{mic}}(\mathbf{V}, T) \equiv P(\kappa; n) \sim \sqrt{n} e^{-n[\kappa - g(\kappa)]}, \quad (2.22)$$

where the function $g(\kappa)$ is called spectrum of entropies, in analogy with $f(\alpha)$ [78,30]. By substituting Eq.(2.22) into Eq.(2.21), the integral can be evaluated by means of a saddle-point approximation, in the limit $n \rightarrow \infty$, as in Section 2.2. This leads to

$$(q-1)K(q) = \inf_{\kappa} \{q\kappa - g(\kappa)\}, \quad (2.23)$$

in complete analogy with Eq.(2.9). The infimum is obtained for $\kappa = \kappa(q) = d[(q-1)K(q)]/dq$. The behaviour of $g(\kappa)$ is the same as that of $f(\alpha)$: the maximum $\max_{\kappa} \{g(\kappa)\}$ is equal to the topological entropy $K(0)$, $g'(\kappa) = q$ and $g[\kappa(1)] = \kappa(1)$, with $\kappa(1)$ being the Kolmogorov-Sinai metric entropy (see Section 1.4). Again, Eq.(2.23) is formally equivalent to the thermodynamical relation (2.11) where, now, $\mathcal{F}(T) = -K(q)$, $\mathcal{E}(T) = -\kappa(q)$ and $\mathcal{S}(\mathcal{E}) = k_B [g(-\mathcal{E}) + \mathcal{E}]$. Finally, the pressure P is equal to $K(q)$. For a hyperbolic system, the metric entropy coincides with the positive Lyapunov exponent, so that the whole procedure can be repeated for the function $h(\Lambda)$ [78], which is defined through the probability density $P(\Lambda; n)$ to observe an effective Lyapunov exponent $\lambda(n) \in [\Lambda, \Lambda + d\Lambda]$ [30]: $P(\Lambda; n) \sim \sqrt{n} e^{-n[\Lambda - h(\Lambda)]}$. All previous relations are valid, with the substitutions: $\kappa \rightarrow \lambda$, $g \rightarrow h$, $K(q) \rightarrow L(q)$, where $L(q)$ is the generalized Lyapunov exponent [43,44]. This approach is essentially that of Refs.[58], apart from the sign of energy and temperature.

In higher dimensional systems, pointwise dimensions, metric entropies and Lyapunov exponents become vector-quantities and the more general relations (1.56) and (1.58) must be used. It is, then, convenient to cover the set with ellipsoids, the axes of which can be expressed in terms of a common length-scale ε [30]:

$$\varepsilon_k = \varepsilon^{\eta_k}, \quad k = 1, \dots, E, \quad (2.24)$$

where the exponents η_k can be constrained in the interval $[0, 1]$, by defining $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_E\}$. The usual definition of the isotropic case ($\varepsilon_k = \varepsilon, \forall k$) is recovered when $\eta_k = 1, \forall k$. By substituting Eq.(2.24) into Eq.(1.45), the crowding index $\alpha(\varepsilon, \mathbf{x})$ can be written as

$$\alpha(\varepsilon, \mathbf{x}) = \frac{\ln P(\vec{\varepsilon}, \mathbf{x})}{\ln \varepsilon} = \vec{\eta} \cdot \vec{\alpha}, \quad (2.25)$$

where $\vec{\alpha} = (\alpha_1, \dots, \alpha_E)$. The average (1.46) can then be evaluated as in Eq.(2.8), introducing a different scaling function $f(\alpha; \vec{\eta})$ for each choice of $\vec{\eta}$ [30]:

$$\alpha(q) = \vec{\eta} \cdot \frac{d\vec{\tau}(q)}{dq}, \quad f(\alpha; \vec{\eta}) = q\alpha(q) - \vec{\eta} \cdot \vec{\tau}(q), \quad (2.26)$$

where $\vec{\tau}(q) = (q-1)\vec{d}(q)$. This implies that both $\alpha(q)$ and $f(\alpha; \vec{\eta})$ are linear in $\vec{\eta}$:

$$\alpha(q) = \vec{\eta} \cdot \vec{\alpha} = \sum_{k=1}^E \eta_k \alpha_k, \quad \text{with } \alpha_k(q) = d\tau_k(q)/dq, \quad (2.27)$$

and $f(\alpha; \vec{\eta}) = \vec{\eta} \cdot \vec{f}(\vec{\alpha})$, with $f_k(\alpha_k) = q\alpha_k(q) - \tau_k(q)$. For a more detailed analysis see Ref.[30], where analogous relations are given also for $g(\kappa)$ and $h(\vec{\Lambda})$. From the point of view of statistical mechanics, this formalism describes a mixture of E substances with partial energies $\mathcal{E}_k(\mathbf{T}) = -\alpha_k(q)$, partial entropies $\mathcal{S}_k(\mathcal{E}) = k_B [f_k(-\mathcal{E}_k) + \mathcal{E}_k]$, partial pressures $P_k = d_k(q)$, and so on.

For non-hyperbolic systems, the metric entropy does not coincide with the sum of the positive Lyapunov exponents (see Eq.(1.59)) and the spectra $g(\kappa)$ and $h(\sum_{k=1}^{j^+} \Lambda)$ are different (see Eq. (1.59)). An example of this discrepancy is provided by the logistic map (1.9) and is discussed in Part III (Section 3.2). However, useful relations among the spectra $f(\alpha)$, $g(\kappa)$ and $h(\vec{\Lambda})$ can be found in special cases. For two-dimensional maps with constant jacobian $|J| \equiv e^{-B} = e^{\lambda_1 + \lambda_2}$, in fact, we have [49,30]

$$\alpha_2(q) = \frac{\lambda_1(q)}{\lambda_1(q) + B}, \quad \text{where } \lambda_1(q) = \frac{d[(q-1)L_1(q)]}{dq}, \quad (2.28)$$

and

$$f_2(\alpha_2)/\alpha_2 = h(\Lambda_1)/\Lambda_1. \quad (2.29)$$

The latter relation holds, more generally, for all $2-d$ maps in which there is a one-to-one correspondence between the positive and the negative Lyapunov exponent (e.g., the baker map (1.44)). Finally, it is worth mentioning that a Grand Canonical

ensemble could also be introduced [62], by allowing the number n of iterations to fluctuate. However, since the conservation laws of the previous sections are written for constant n , it is not necessary to examine this case, here. A variable n can be obtained in many ways. For instance, by taking a fixed-mass covering in trajectory space, a position-dependent $n = n(\mathbf{x})$ would follow, as a result of the fluctuations of the pointwise entropy $\kappa(\mathbf{x})$ (see Eq.(1.56)), since the mass scales like $P(\vec{\varepsilon}, X(n)) \sim e^{-n\kappa(\mathbf{x}, n)}$, for large n . In Ref.[62], $n = n(\mathbf{x}, \varepsilon)$ was defined as the first return time in an ε -neighbourhood of point $\mathbf{x}$.

Part III

Phase-Transition-like Behaviour in Dynamical Systems

As discussed in Part II, it is possible to map nonlinear dynamical systems onto suitable hamiltonian spin-models on a one-dimensional lattice. This formal correspondence finds its major justification in the existence of phenomena which can be described as phase-transitions. They occur for non-self-similar sets and are due to long-time correlations in the symbolic dynamics: this renders the scaling function (2.18) dependent on the whole "history" $\{\dots, s_{-n}, \dots, s_{-1}, s_0\}$ and not just on the last (few) symbols. Such non-self-similarities can be created in various ways: in Section 3.1, we give an example of a 0-th order transition (discontinuity in the dimension function) arising from the union of different Cantor sets. In Section 3.2, two non-hyperbolic systems (logistic and Hénon maps, Eqs.(1.9) and (1.3)) are analyzed, as typical examples of generic physical systems. The results on these maps suggest that non-hyperbolicity is a source of long-range memory effects in the dynamics. However, almost no theoretical results are available for such transformations, at variance with hyperbolic systems. The advantage in studying the latter ones comes from the possibility of finding generating partitions (see Section 1.4) and, hence, of constructing the symbolic dynamical representation analytically. The main result of this Part is that non-self-similarities and, consequently, phase-transition-like behaviour can also be generated in fully hyperbolic systems. In Sections 3.3 and 3.4, the filtered baker map is introduced and solved analytically, showing that two different transitions occur. The first can be described in purely geometric terms and corresponds to the passage from a self-similar to a self-affine structure; the second is related to the probabilistic features of the attractor. Such phenomena are caused, in non-hyperbolic systems, by the homoclinic tangencies (i.e., between stable and unstable manifolds) while, in the hyperbolic case, they arise from the occurrence of tangencies between different eigen-directions in the contracting (i.e., stable) sub-space. Therefore, the lowest dimensionality necessary to observe them in a map is two, for non-hyperbolic systems, and three, for hyperbolic ones.

3.1 Countable Union of Cantor Sets

The simplest way to produce a non-self-similar fractal is to take a (countable) union of uniform Cantor sets with different dimensions. To be more specific, let us consider M of such sets, with dimensions D_j ($j = 1, \dots, M$), and construct a new fractal by choosing points at random from them, with the attribution of the probability weight p_j to the j -th set ($\sum_{j=1}^M p_j = 1$). Performing, then, the average (1.24) (fixed-mass method), we obtain (the ball-size scales as $(k/n)^{1/D_j}$, in the j -th set)

$$\sum_{j=1}^M p_j \left(\frac{k}{n} \right)^{(1-q)D_j/D_j} \sim \left(\frac{k}{n} \right)^{1-q}, \text{ for } k/n \rightarrow 0, \quad (3.1)$$

where k is the number of points in each ball of the covering and n is the total number of points (k/n approximates the mass in the ball). In the limit $n \rightarrow \infty$

(for fixed k), the dimension function $D(q)$ is found:

$$D(q) = \begin{cases} \min_j \{D_j\} & \equiv D_+ \quad \text{for } q > 1, \\ 1 / \sum_{j=1}^M (p_j / D_j) & \text{for } q = 1, \\ \max_j \{D_j\} & \equiv D_- \quad \text{for } q < 1. \end{cases} \quad (3.2)$$

The plot of $D(q)$ versus q is displayed in Fig. 12. For q larger than 1, the “phase”

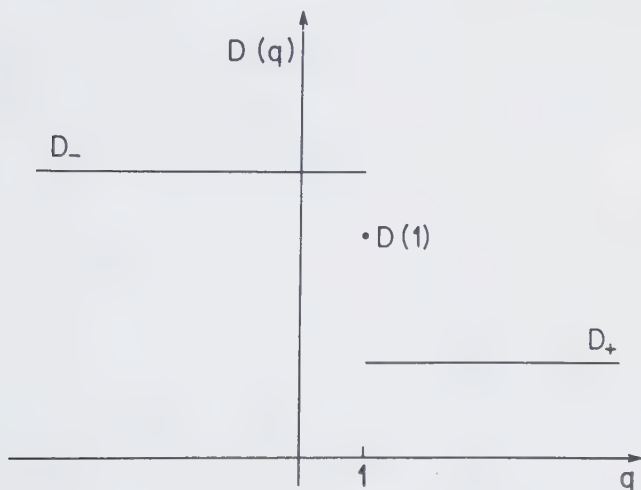


Figure 12: Plot of $D(q)$ vs. q , for a countable union of different Cantor sets.

consisting of the set with the smallest dimension ($D(q) = D_+$) predominates while, for q smaller than 1, the opposite situation occurs ($D(q) = D_-$). The transition point is at $q = 1$, where the “thermodynamic” function $D(q)$ itself is discontinuous. This is analogous to a 0-th order phase-transition. Notice that the isolated point $D(q = 1)$ can assume all values between D_+ and D_- , according to the choice of the probability weights p_j .

The dimension spectrum $f(\alpha)$ has the form

$$f(\alpha) = \begin{cases} -\infty & \text{if } \alpha \notin [D_+, D_-], \\ \alpha & \text{if } \alpha \in [D_+, D_-] \end{cases} \quad (3.3)$$

and is displayed in Fig. 13. In the case of the union of only two Cantor sets ($M = 2$), a direct evaluation of $f(\alpha)$ from the probability distribution (2.6) would yield

two vertical segments, at $\alpha = D_+$ and $\alpha = D_-$, terminating at the bisectrix $f(\alpha) = \alpha$, without the linear part present in Fig. 13. For a generic M , we would observe M vertical segments: indeed, we have different “phases”, supported by sets of different dimensions. The line between D_+ and D_- is obtained by computing $f(\alpha)$ from $D(q)$, through the Legendre transform (2.9). Of course, similar considerations

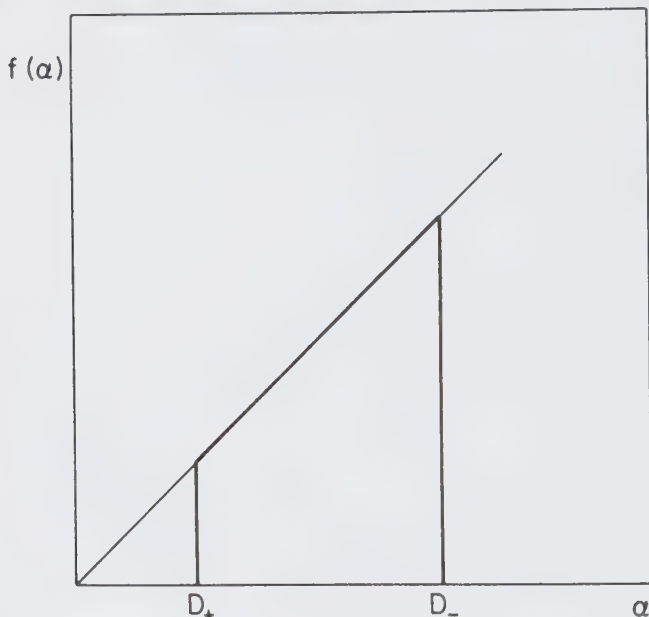


Figure 13: Plot of $f(\alpha)$ vs. α for the same example as in Fig. 12.

apply to the union of two (or more) non-uniform Cantor sets, as well. In that case, the function $f(\alpha)$ would present two (or more) humps, if directly computed from the histogram (Eq.(2.6)), and would not be, therefore, concave. However, the evaluation of $D(q)$ and the application of relation (2.9) would yield a concave $f(\alpha)$, displaying a linear piece connecting the two humps (and tangent to them), in analogy with Maxwell’s rule (Section 2.3).

3.2 Non-hyperbolic Systems: the Logistic and Hénon maps

Although the model of the previous section is rather unphysical, it provides a first example of the possible phenomenology in nonlinear dynamics: indeed, it can be considered as the solution of a set of uncoupled baker maps, together with a suitable sampling rule (for the choice of the p_j 's).

In this section, we study two non-hyperbolic systems that are often taken as models of generic physical situations (real systems are, usually, non-hyperbolic). The first one is the logistic equation (1.9), which can be exactly solved in the case of “fully developed” chaos, corresponding to the parameter value $a = 2$. The invariant measure, given in Eq.(1.10), is computed through the conjugacy to the symmetric tent map (Eq.(1.43) with $p_1 = p_2 = 1/2$), with the substitution $x = \sin(\pi y/2)$ [18]. By covering the segment $(-1, 1)$ with boxes of size ε , the mass in the leftmost and in the rightmost ones is proportional to $\sqrt{\varepsilon}$ while, in all others, it scales like ε . Hence, we have

$$\sum_{j=1}^{1/\varepsilon} P_j^q(\varepsilon) \sim \varepsilon^{q-1} + c \cdot \varepsilon^{q/2}, \quad (3.4)$$

where c is a constant. From the definition of dimension function (1.15), we obtain [79]

$$D(q) = \begin{cases} 1 & \text{if } q \leq 2, \\ \frac{q}{2(q-1)} & \text{if } q \geq 2. \end{cases} \quad (3.5)$$

That is, $D(q)$ is not unity for all q , in contrast to what happens in hyperbolic systems. In this case, there is a discontinuity in the derivative of the “free energy” $D(q)$ at $q = 2$. This singular point is transformed, through Eq.(2.9), into a segment with slope $f'(\alpha) = q = 2$, in the graph of $f(\alpha)$:

$$f(\alpha) = \begin{cases} 2\alpha - 1 & \text{for } \frac{1}{2} \leq \alpha \leq 1, \\ -\infty & \text{otherwise.} \end{cases} \quad (3.6)$$

Notice that Eq.(2.10) would have only predicted $f(1/2) = 0$ and $f(1) = 1$, without the linear behaviour in between, which can also be obtained with a direct evaluation of $f(\alpha)$ from the histogram of the probability distribution (2.6) (see, later, Fig. 14).

The non-analytic behaviour of $D(q)$ at $q = q_c = 2$ was interpreted, in Ref.[61], as a phase-transition at negative temperature T for a spin-system with anti-ferromagnetic two-spin interactions (or, by reverting the sign of T , with ferromagnetic interactions). With the definitions of Eqs.(2.15), the choice of temperature and energy coincides with the latter case. The states characterized by the spin values $(-\frac{1}{2}, -\frac{1}{2}, \dots)$ (fixed point $x = -1$) and $(\frac{1}{2}, \frac{1}{2}, \dots)$ (point $x = 1$) are the ground states. The hamiltonian contains interaction terms between $2, 3, \dots, n$ spins and all orders of nearest-neighbours, with amplitudes that are essentially independent of

the distance [61]. This behaviour is a consequence of the evident non-self-similarity of the set: the probabilities scale like ε everywhere except at the borders of the interval, where they are proportional to $\sqrt{\varepsilon}$. The square-root singularities in the measure are generated by the strong mass compression around $x = 0$ (i.e., in the non-hyperbolic region of the map).

Differently from what happens for the dimension function $D(q)$, no singularities are expected for metric entropies. In fact, due to the conjugacy to the symmetric tent map, the generalized entropies are $K(q) = \ln 2$, for any q . The spectrum of Lyapunov exponents $h(\Lambda)$, instead, is not as trivial as $g(\kappa)$. For $-\infty < \Lambda \leq \ln 2$, it displays a straight-line behaviour with slope 2; it is constant (equal to $\ln 2$) for $\ln 2 \leq \Lambda \leq \ln 4$ and is $-\infty$ for $\Lambda > \ln 4$ [30]. Indeed, effective Lyapunov exponents assume all values between $-\infty$ and $\ln 4$. The former value corresponds to orbits which pass in the vicinity of the critical point $x = 0$, and the latter to the fixed point at $x = -1$. Notice that a different result would have been obtained if we had considered the spectrum $\tilde{h}(\Lambda)$ for periodic orbits of period n (the approach of Refs.[58],[80],[81]) and not randomly-chosen trajectories: in fact, all cycles yield $\lambda = \ln 2$, except for the fixed point $x = -1$, where $\lambda = \ln 4$. In particular, the spectrum of Lyapunov exponents of periodic points cannot extend to negative values (as it does for non-periodic orbits) since, otherwise, we would have attracting cycles and no chaotic attractor. The two spectra $h(\Lambda)$ and $\tilde{h}(\Lambda)$ coincide for hyperbolic maps (and, they also coincide with $g(\kappa)$ since the dimension is, in that case, everywhere equal to 1).

Similar results can be found in the more generic case in which the critical point does not map onto a periodic orbit: then, square-root singularities appear in the natural measure at all forward images of the maximum (see Fig.4). This situation is called “typical” chaos, since it appears on a set of control parameter values of non-zero measure. Being the singularities denumerable, their effect on $D(q)$ and $f(\alpha)$ is the same as in the case $a = 2$. The result of a numerical evaluation of $f(\alpha)$, for $a = 1.85$, is presented in Fig. 14 (from [30]). The entropy spectrum $g(\kappa)$ is in good agreement with that of Lyapunov exponents $h(\Lambda)$, for $\Lambda > 0.3$. Again, differences are found for $\Lambda < \lambda_{\min}$, where $\lambda_{\min} (> 0)$ is the smallest Lyapunov exponent of any periodic trajectory (for $a = 1.85$, a $\lambda_{\min}$ around 0.237 was found, corresponding to a period-11 cycle, out of all periodic orbits up to a maximum period of 33). The spectrum $g(\kappa)$ (computable from the symbol-sequences) is, actually, well approximated by $h(\Lambda)$. Since the singularity found in the dimension function $D(q)$ is related to the tail $\lambda < \lambda_{\min}$, it is clear that no analogous behaviour can occur for $K(q)$.

The non-self-similarity, which manifests itself in the presence of square-root singularities in the natural measure, is therefore due to contraction of distances (i.e., mass compression) caused by the region around the maximum, where the slope of the map is, in modulus, less than 1. This is also the source of slowly-decaying correlations in the symbolic dynamics. In fact, in hyperbolic systems, the asymptotic time-behaviour of correlation functions $C(n)$ is given by the average inverse multiplier: $C(n) \sim \langle \mu^{-1}(n) \rangle$ [82]. In non-hyperbolic maps, $|\mu(n)|$ can be

smaller than 1, so that the above relation for $C(n)$ does not hold and slower-than-exponential decay may occur.

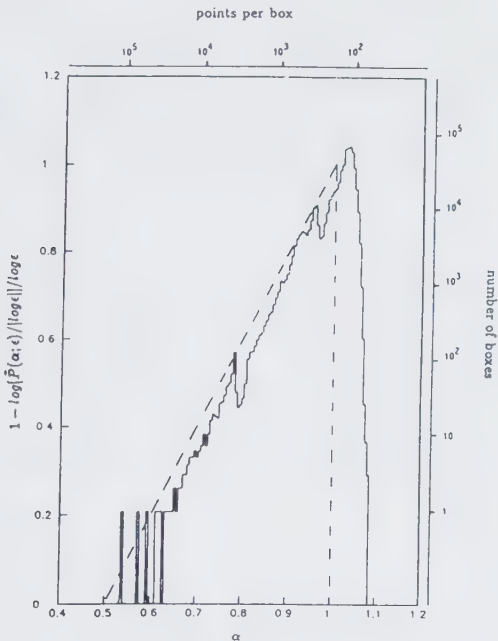


Figure 14: Histogram of the distribution of the number of boxes with a given weight, for the logistic map at $a = 1.85$. The data are displayed on a log-log plot in such a way that the curve $f(\alpha)$ vs. α is obtained, in the limit $\epsilon \rightarrow 0$. The dashed line represents Eq.(3.6).

The Hénon map (1.3) is a typical example of a two-dimensional non-hyperbolic transformation. Similarly to the previous 1- d case, the invariant measure presents square-root singularities at the homoclinic tangency points (see Section 1.5) which are, in this respect, the 2- d counterpart of the critical point $x = 0$ of the logistic map. Therefore, we expect the Hénon transformation to behave essentially as the latter, with minor modifications due to its two-dimensionality (it actually reduces to the logistic equation, for $b \rightarrow 0$). In the neighbourhood of a homoclinic tangency, the map can be schematized as follows: the unstable manifold is represented by a Cantor set of parabolas and the stable manifold by a dense set of parallel straight lines (see Fig. 1). The action of the map is approximated by a linear transformation with a fixed contraction rate γ_2 along the stable manifold and an expansion rate

$\gamma_1 = b/\gamma_2$, transverse to it.

A direct estimate of $f(\alpha)$ was obtained in Ref.[30] from the definition (2.6), by following a box-counting procedure. In Fig. 15, the histogram of the distribution of box weights is displayed in such a way that the plot tends to the curve $f(\alpha)$ vs. α , for $\varepsilon \rightarrow 0$ and for $\alpha \leq \alpha(q = 0)$ (corresponding to positive q : for $q < 0$, box-counting is affected by large statistical errors). Two coverings were used: the first one consisting of 14336^2 boxes (dotted line) and the second one of 28672^2 boxes (solid line). The “standard” parameter values $a = 1.4$ and $b = 0.3$ have been chosen [15]. The number of iterations was $4 \cdot 10^9$ and the number of non-empty boxes was $\approx 9 \cdot 10^5$.

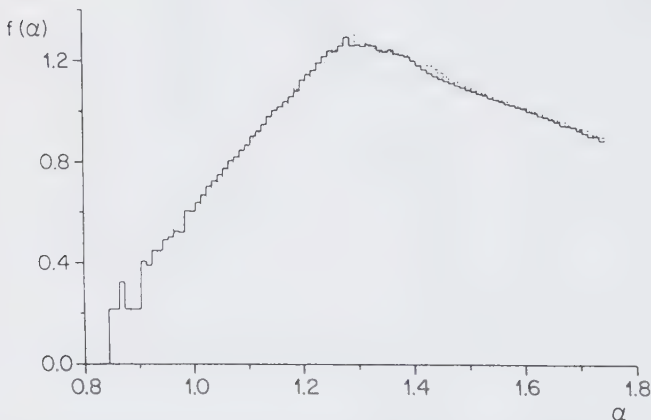


Figure 15: Estimate of the curve $f(\alpha)$ vs. α for the Hénon map, with parameters $a = 1.4$ and $b = 0.3$, obtained with a box-counting method. Solid line: grid containing $28762 \cdot 28762$ boxes; dotted line: $14336 \cdot 14336$ boxes. The curves converge to $f(\alpha)$, in the limit $\varepsilon \rightarrow 0$.

The minimal α at which $f(\alpha) \neq -\infty$ is $\alpha_{\min} = \lim_{q \rightarrow \infty} D(q) \leq 0.84$. Since this value is smaller than 1, the first partial dimension $d_1(q)$ is smaller than 1, for large q (while it is equal to 1, for small q). Hence, there must be a singularity in the derivative of $D(q)$, at some point $q = q_c$ and the function $f(\alpha)$ must exhibit a linear portion with slope $f'(\alpha) = q_c$. Indeed, such a behaviour is evident in the figure, from which the critical “inverse temperature” q_c can be estimated to be $q_c \approx 2.4$.

The scaling function $h(\Lambda_1)$ for the first Lyapunov exponent is similar to that obtained for the logistic map (here, since the jacobian $J = -b = \Lambda_1 + \Lambda_2$ is constant, we have only one independent Lyapunov exponent). Again, the first

effective Lyapunov exponent can assume negative values, for any finite n (with exponentially decreasing probability, in the limit $n \rightarrow \infty$). Hence, as in the $1-d$ case, $h(\Lambda_1)$ cannot agree with the spectrum $\tilde{h}(\Lambda_1)$ of periodic orbits. Small values of λ_1 arise from trajectories which pass near to some homoclinic tangency. At variance with the $1-d$ case, the curve $h(\Lambda_1)$ has now a cut-off at $\Lambda_1 = (\ln b)/2 \approx -0.602$ [30].

The results of the two examples of this section can be interpreted as follows: for natural measures on expanding directions of non-hyperbolic systems, the averages (1.22) or (1.50) are dominated by the “normal” (hyperbolic) points for $q < q_c$, and by homoclinic tangencies (critical point, for $1-d$ maps) for $q > q_c$. The situation is analogous to the coexistence of two different phases, manifesting itself in the non-analytic behaviour of $D(q)$. No such singularities are expected for the metric entropies $K(q)$ since they are, essentially, well reproduced by the spectrum $\tilde{h}(\Lambda_1)$ of Lyapunov exponents of periodic orbits, which are all unstable (i.e., hyperbolic). For $q \leq q_c$, relation (1.52) (which was derived for hyperbolic systems) is, therefore, expected to relate the partial dimensions $d_k(q)$ to the Lyapunov exponents in a correct way. For $q > q_c$, instead, it is no more applicable, since the “phase” consisting of the non-hyperbolic points prevails. In Ref. [30], this relation has been used to yield a more accurate estimate of q_c than that obtainable from box-counting: the result for the Hénon map is $q_c \approx 2.25$. An alternative approach, based on the Lyapunov exponents of periodic points, has been recently proposed [80,81].

3.3 Filtered Baker map: Uniform Case

From the discussion of the previous two sections, it is clear that non-analytic behaviour in the dimension function $D(q)$ arises from a lack of self-similarity, which occurs if two or more regions with different scaling properties (“phases”) exist. By varying the parameter q , their relative weight in the averages yielding $D(q)$ can be controlled. From a dynamical point of view, this situation corresponds to long-memory effects in the scaling function (2.18) which, thus, depends on the whole semi-infinite symbolic sequence $S_{-\infty} = \{\dots, s_{-1}, s_0\}$. In non-hyperbolic systems, such a behaviour originates from the possibility that effective Lyapunov exponents assume negative values, along the expanding directions, for any large (but finite) number n of iterates. In fact, a positive Lyapunov exponent λ_1 implies exponential amplification of the uncertainty on the initial condition (i.e., loss of memory), while negative λ_1 -values determine slowly-decreasing correlations [82].

However, non-hyperbolicity is not a strictly necessary (although sufficient) condition to produce fractals without asymptotic self-similarity. Indeed, we introduce, in this section, a three-dimensional hyperbolic map exhibiting a “phase-transition-like” behaviour, under the control of an external parameter. This is obtained through the memory effect induced by a low-pass filtering action on the well-known baker transformation (1.44). The advantage of working with a piece-wise linear hyperbolic map is that the symbolic-dynamical representation of all variables can be found exactly, thus allowing for an analytic solution of the problem.

Consider an experiment on a non-linear system (e.g., a laser or a fluid in a Raleigh-Bénard cell) and measure the output signal $x(t)$ (a single scalar time-series). Usually, the system can be modelled through a set of differential equations of the type (1.1) although, in general, their exact form is unknown. Therefore, for the analysis of the data, it is necessary to reconstruct the underlying attractor in an embedding space (see Section 1.4) of suitably high dimension [83]. In addition, the signal $x(t)$ is often filtered, in order to reduce the effect of noise and possible spurious high-frequency components. The analysis is, then, performed on the output $z(t)$ of the filter and not on the original $x(t)$. This procedure, however, can alter the estimate of the fractal dimension: i.e., the dimension of the filtered signal $z(t)$ may be larger than that of the system’s output $x(t)$, depending on the filter parameter [77]. This phenomenon is responsible for the appearance of a non-self-similar structure.

Assume, for simplicity, that only three Lyapunov exponents $\langle \lambda_1 \rangle > 0, \langle \lambda_3 \rangle < 0$ determine the information dimension of the system, through the Kaplan-Yorke relation (see Section 1.6): $D(1) = 2 + \langle \lambda_1 \rangle / |\langle \lambda_3 \rangle|$. This value would be the result of a direct computation of $D(1)$ performed on the output $x(t)$ of the experiment, in the embedding space. We then process the signal $x(t)$ with a single-pole low-pass filter, whose dynamics can be described by the additional linear differential equation

$$\dot{z} = -\eta z + x, \quad (3.7)$$

where $z(t)$ is the output of the filter and $\eta > 0$ is the cut-off frequency. Therefore,

the dynamical system associated to the variable z consists of the original (unknown) equations yielding $x(t)$ together with Eq.(3.7). As a result, since equation (3.7) does not affect the previous ones, the Lyapunov exponents of the new system are obtained just by adding $-\eta$ to the original list. The relevant exponents for the

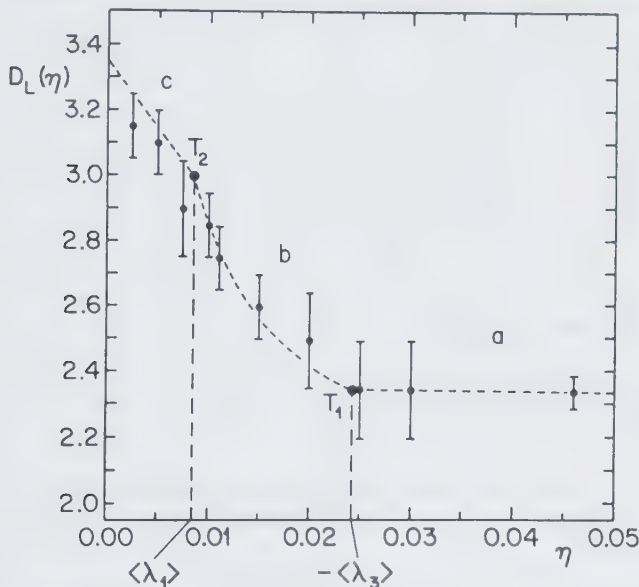


Figure 16: Plot of the Lyapunov dimension $D_L(\eta)$ vs. η , for a filtered chaotic signal (NMR-laser, see text). The directly estimated values of the information dimension $D(1)$ are indicated with the corresponding error bars.

determination of the Lyapunov dimension $D_L = D(1)$ are, then,

$$\{(\lambda_1), 0, (\lambda_3), -\eta\}, \quad (3.8)$$

where the order depends on the magnitude of η . In particular, the Lyapunov dimension $D_L(\eta)$ exhibits, as a function of η , the following behaviour (corresponding to the three regions a, b and c displayed in Fig. 16):

$$D_L(\eta) = \begin{cases} 2 + \langle \lambda_1 \rangle / |\langle \lambda_3 \rangle| & , \text{ for } -\eta < \langle \lambda_3 \rangle, \\ 2 + \langle \lambda_1 \rangle / \eta & , \text{ for } \langle \lambda_3 \rangle < -\eta < -\langle \lambda_1 \rangle, \\ 3 + ((\lambda_1) - \eta) / |\langle \lambda_3 \rangle| & , \text{ for } -\langle \lambda_1 \rangle < -\eta < 0. \end{cases} \quad (3.9)$$

In the first region, corresponding to large η , the variable z can be adiabatically eliminated and coincides with x , apart from a scale-factor. When the value $-\eta$

increases beyond $\langle \lambda_3 \rangle$, it becomes the third Lyapunov exponent in the list and takes the place of $\langle \lambda_3 \rangle$ in the expression of D_L . Finally, in the third region, η is so small that $\langle \lambda_1 \rangle - \eta$ is still non-negative and the dimension becomes larger than 3, increasing linearly for $0 < \eta < \langle \lambda_1 \rangle$. The largest increment (equal to 1) is obtained in the degenerate case $\eta = 0$, when $z(t)$ is no longer a stationary process. In fact, $z(t)$ reduces to the integral of the chaotic signal $x(t)$ whose power spectrum $S(\omega)$, usually, does not vanish faster than ω , for $\omega \rightarrow 0$. The prediction (3.9) has been numerically verified, in Ref.[77], for the forced Duffing equation and, in Ref.[84], for two experimental systems (Raleigh-Bénard convection and NMR-laser): the direct estimates of $D(1)$ for the latter case are reported in Fig. 16. In the present work, however, the focus is on the analogy with phase-transitions, rather than on the increase of the dimension when the bandwidth η decreases.

To this end, we consider the generalized baker transformation (1.44) as representative of the “experimental” system. In order to write the discrete analogue of Eq.(3.7), it is necessary to choose an appropriate embedding variable (the output of the system) which allows to obtain a faithful reconstruction of the attractor. In the case of the baker map, neither x nor y can be used for this purpose: in fact, the variable y is decoupled from x , while the evolution of x cannot be inferred from the knowledge of its own past history alone. Therefore, it is convenient to consider a linear combination of the two variables: $w_n = x_n + y_n$. The equation for the filter is, then,

$$z_{n+1} = \gamma z_n + w_{n+1}, \quad (3.10)$$

where $0 \leq \gamma < 1$ is the filter constant: the associated Lyapunov exponent is $\ln \gamma$ and plays the role of $-\eta$ (Eq.(3.7)). By introducing the symbolic representation (see Section 1.6)

$$s_{-n} \equiv \frac{1 + \text{sgn}(y_n - p_1)}{2}, \quad (3.11)$$

it is possible to rewrite Eq.(1.44) in the form

$$\begin{aligned} x_n &= x_{n-1} r_1 \left(\frac{r_2}{r_1} \right)^{s_{-n+1}} + (1 - r_2) s_{-n+1} \\ y_n &= y_{n+1} p_1 \left(\frac{p_2}{p_1} \right)^{s_{-n}} + p_1 s_{-n}, \end{aligned} \quad (3.12)$$

where y is iterated backwards for reasons of convergence. Iteration of Eqs.(3.12) allows to write expansions for the variables x and y in the form

$$x_n = (1 - r_2) \left[s_{-n+1} + \sum_{i=1}^{\infty} s_{-n+i+1} r_1^i \prod_{j=1}^i \theta^{s_{-n+j}} \right] \quad (3.13)$$

$$y_n = p_1 \left[s_{-n} + \sum_{i=1}^{\infty} s_{-n-i} p_1^i \prod_{j=1}^i \pi^{s_{-n-j+1}} \right], \quad (3.14)$$

where the ratios $\theta = r_2/r_1$ and $\pi = p_2/p_1$ have been introduced. These expressions can be simplified by making use of the symmetry properties of the baker map. The

invariance of Eq.(1.44) under the transformation

$$\begin{aligned} x &\longrightarrow 1 - x \\ r_1 &\longleftrightarrow r_2 \\ s_{-n} &\longrightarrow 1 - s_{-n} , \end{aligned} \quad (3.15)$$

allows to rewrite Eq.(3.13) as

$$x_n = \frac{1 - r_2}{r_2 - r_1} \left[-r_1 + (1 - r_1) \sum_{i=1}^{\infty} r_1^i \prod_{j=1}^i \theta^{s_{-n+j}} \right]. \quad (3.16)$$

Similarly, for the y -variable, the invariance of Eq.(3.14) under the transformations

$$\begin{aligned} y &\longrightarrow 1 - y \\ p_1 &\longleftrightarrow p_2 \\ s_{-n} &\longrightarrow 1 - s_{-n} \end{aligned} \quad (3.17)$$

leads to

$$y_n = \frac{p_1}{p_2 - p_1} \left[-p_1 + p_2 \sum_{i=1}^{\infty} p_1^i \prod_{j=1}^i \pi^{s_{-n-j+1}} \right]. \quad (3.18)$$

The formal solution of the filter's equation (3.10) yields, finally, the expression for z_n as

$$z_n = \sum_{k=0}^{\infty} w_{n-k} \gamma^k = \sum_{k=0}^{\infty} (x_{n-k} + y_{n-k}) \gamma^k. \quad (3.19)$$

In this section, we restrict the analysis to the uniform case $r_1 = r_2 = r$ and $p_1 = p_2 = 1/2$, delaying the complete treatment to the next section. With this simplification, the substitution of relations (3.13) and (3.14) into (3.19) yields

$$z_n = \sum_{k=0}^{\infty} \gamma^k \left[(1 - r) \sum_{i=0}^{\infty} s_{-n+k+i+1} r^i + \frac{1}{2} \sum_{i=0}^{\infty} s_{-n+k-i} 2^{-i} \right]. \quad (3.20)$$

Defining $j = k + i + 1$ and $l = k - i$, we obtain

$$z_n = \frac{1 - r}{r - \gamma} \sum_{j=1}^{\infty} s_{-n+j} (r^j - \gamma^j) + \frac{1}{2 - \gamma} \left[\sum_{l=1}^{\infty} s_{-n+l} \gamma^l + \sum_{l=0}^{-\infty} s_{-n+l} 2^l \right]. \quad (3.21)$$

Since the attractor is continuous along the expanding direction (variable y), we can take a Poincaré section by cutting it with the plane $y_n = 0$. This condition corresponds to setting $s_{-i} = 0$ for $i \geq n$, as it can be checked in Eq.(3.14). In this way, the third sum in Eq.(3.21) is identically zero and we, finally, have

$$z_n = \sum_{j=1}^{\infty} s_{-n+j} \left\{ \frac{1 - r}{r - \gamma} r^j + \left(\frac{1}{2 - \gamma} - \frac{1 - r}{r - \gamma} \right) \gamma^j \right\}. \quad (3.22)$$

It is, therefore, evident that the asymptotic behaviour of z is either dominated by the contraction rate τ , if $\tau > \gamma$ (i.e., weak filtering action), or by γ , if $\tau < \gamma$. In the former case, the variable z is “proportional” to x and the asymptotic structure is self-similar; in the latter one, every subset of the attractor can be rescaled to the previous level, provided that the distances in the z -direction are multiplied by $1/\gamma$ and those in the x -direction by $1/\tau$: that is, the set is asymptotically self-affine (see Section 1.6). The transition at $\gamma = \tau$ (point T_1 in Fig. 16) can be

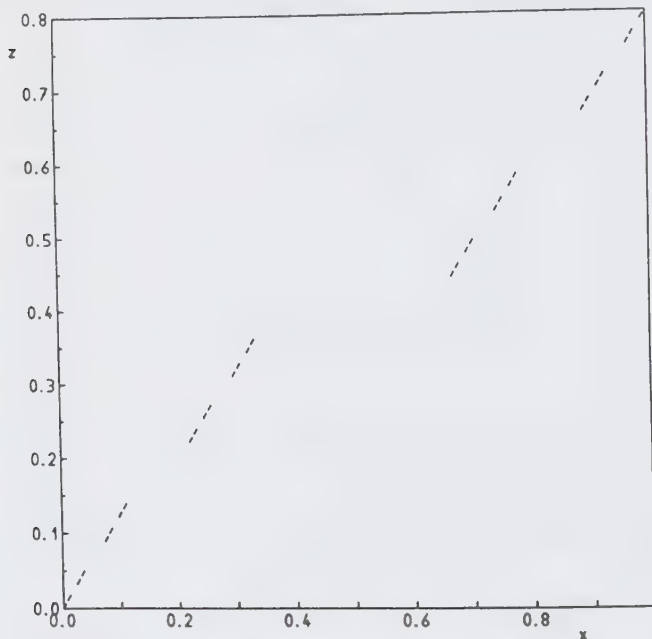


Figure 17: Section of the attractor of the filtered baker map, with $\tau_1 = \tau_2 = 1/3$ and $p_1 = p_2 = 1/2$, for $\gamma = 0.2$. The z -variable has been rescaled to yield a square picture.

recognized as a typical crossover effect characterizing the beginning of the increase of the dimension. The second transition occurs at the value $\gamma = 1/2$ (point T_2 in Fig. 16), when the projection of the attractor’s section onto the z -axis becomes continuous. The value $\gamma = 1/2$ is found by requiring the uppermost point of a generic subset to have the same ordinate z as the lowermost point of the right-contiguous subset, at the same scale. This condition is simultaneously met at all length scales, by varying γ . The structure of the attractor is displayed in figures 17-19. In Fig. 17, the value of γ is smaller than $\tau = 1/3$ and the attractor is, asymptotically, a slanted Cantor set with the same dimension as in the case $\gamma = 0$

(no filter). In region b (Fig. 16), the variables x and z scale with different rates (r and γ , respectively) and the set is self-affine. At the second transition point

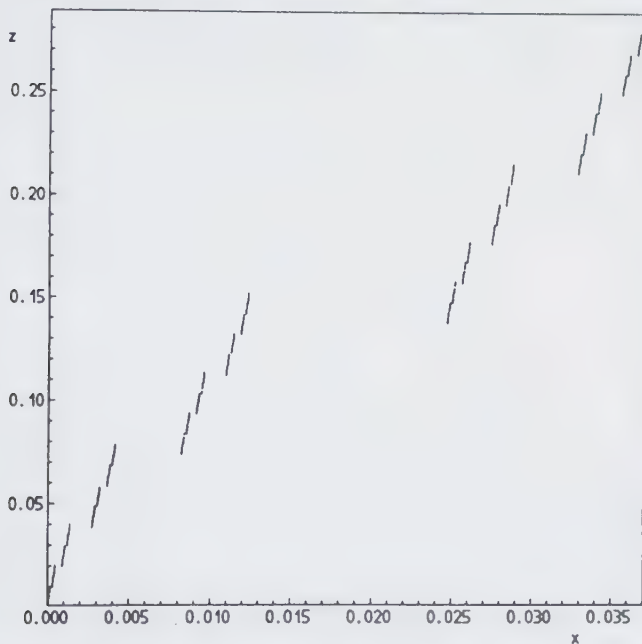


Figure 18: Same as in Fig. 17, for $\gamma = 1/2$ corresponding to the second transition point T_2 .

T_2 ($\gamma = 1/2$), the attractor's section consists of the set of discontinuity points of the Devil's staircase (see Figs. 18 and 6) and its projection onto the z -axis is continuous: its dimension is $D_L(\gamma) = 1$. A further increase of γ breaks the one-to-one correspondence between points in the section and points on the z -axis (the projection is no more single-valued) and the dimension becomes larger than 1 (see Fig. 19). The Kaplan-Yorke relation $D_L = D(1)$ holds for any value of γ .

Since the attractor is uniform, the dimension function $D(q)$ is constant and no phase-transition behaviour occurs. In the non-uniform case, instead, we expect $D(q)$ to behave similarly to the information dimension $D(1)$, for every fixed q , as a function of γ . The increase for lower and lower filter bandwidth $\eta = -\ln \gamma$ will present corner-points located at different γ -values, for different q 's, thus generating non-analytic behaviour of the curve $D(q)$ vs. q . This will be the subject of the next section.

Finally, notice that more complicated kinds of filter could be considered. In the case of an order- k filter, it is necessary to compare the real parts of the k poles with

the sequence of Lyapunov exponents. As a result, a piece-wise differentiable function is obtained, analogously to Eq.(3.9), where more than three distinct regions are present.

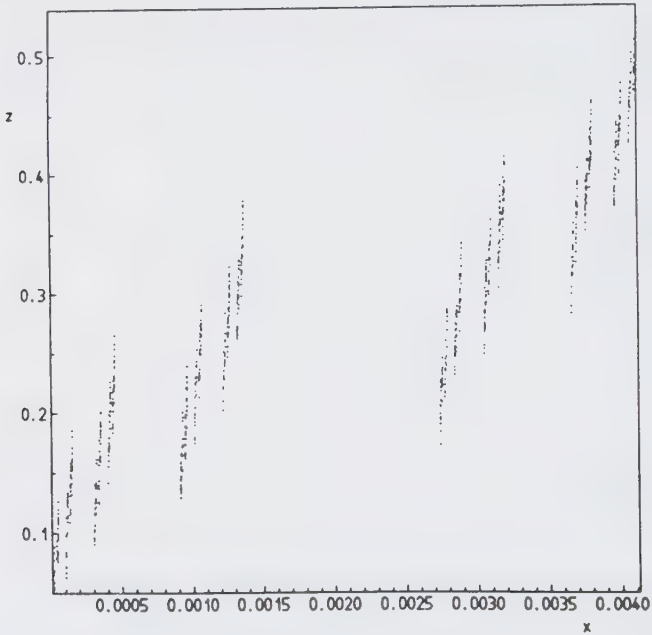


Figure 19: Same as in Fig. 17, for $\gamma = 0.7$ (region c of Fig. 16). The projection of the set onto the z -axis is no more one-to-one and the dimension is larger than 1.

3.4 Non-uniform Case

In this section, we perform a detailed analysis of the first transition (point T_1) for the general non-uniform case and discuss the behaviour of the system in the second critical region (around T_2). By substituting the “bit” expansions for x_n and y_n (Eqs.(3.16) and (3.18)) into Eq.(3.19), we obtain an expression for a generic coordinate z as

$$z = A \sum_{k=0}^{\infty} \gamma^k + B \sum_{k=0}^{\infty} \gamma^k \sum_{i=1}^{\infty} r_1^i \prod_{j=1}^i \theta^{s_{k+j}} + C \sum_{k=0}^{\infty} \gamma^k \sum_{i=1}^{\infty} p_1^i \prod_{j=1}^i \pi^{s_{k-j+1}}, \quad (3.23)$$

where

$$\begin{aligned} A &= \frac{r_1(1-r_2)}{r_1-r_2} + \frac{p^2}{p_1-p_2}, \\ B &= \frac{(1-r_1)(1-r_2)}{r_2-r_1}, \\ C &= \frac{p_1 p_2}{p_2-p_1}, \end{aligned} \quad (3.24)$$

and the index n has been dropped, by shifting the time origin by n units. Accordingly, the Poincaré section at $y = 0$ corresponds, now, to setting $s_i = 0$ for $i \leq 0$.

In order to apply the formalism developed in Part II, it is necessary to evaluate the lengths of the segments which cover the set, at a given level of resolution (the masses are known, since they are not affected by the filter). A projection of the set onto the x -axis yields the non-uniform Cantor set, introduced in Section 1.3, for which all quantities of interest have already been computed. Therefore, we have to evaluate the lengths of the subintervals in the z -direction. This is done by taking the difference

$$\varepsilon(S_n) \equiv z(S_n \cup \{1, 1, \dots\}) - z(S_n \cup \{0, 0, \dots\}) \quad (3.25)$$

between the two extrema of a segment at the n -th level of resolution: the two z -values are characterized by the same symbolic subsequence S_n , for the first n steps, and by the semi-infinite sequences $\{0, 0, \dots\}$ (lower extremum) and $\{1, 1, \dots\}$ (upper extremum), in the asymptotic limit. The first sum in Eq.(3.23) represents a constant shift in z and cancels out in the difference (3.25). The second and third sums are the contributions to z deriving from the expressions for x and y , respectively. We write, therefore, the generic distance (3.25) as $\varepsilon(S_n) = \varepsilon_x(S_n) + \varepsilon_y(S_n)$ and proceed evaluating, first, the term $\varepsilon_x(S_n)$. Notice that the minimum value of the index $k+j$ labelling the exponent s of θ is equal to $k+1$ (for a given k) and that, for $k > n$, all the symbols s_{k+j} are either equal to 1 or to 0. Hence, it is useful to split the sum over k into two parts:

$$\varepsilon_x(S_n) = B \left\{ \sum_{k=0}^{n-1} \gamma^k \sum_{i=1}^{\infty} r_1^i \left[\prod_{j=1}^i \theta^{s'_{k+j}} - \prod_{j=1}^i \theta^{s_{k+j}} \right] + \sum_{k=n}^{\infty} \gamma^k \sum_{i=1}^{\infty} (r_2^i - r_1^i) \right\}, \quad (3.26)$$

where the prime indicates the sequence $S_n \cup \{1, 1, \dots\}$, corresponding to the upper extremum of the segment (the unprimed sequence is, of course, $S_n \cup \{0, 0, \dots\}$). If the largest value assumed by $k + j$ in Eq.(3.26) (equal to $k + i$) is smaller or equal to n , the symbols s and s' coincide and the difference is identically zero. Therefore, we obtain

$$\varepsilon_x(S_n) = \frac{\gamma^n}{1 - \gamma} + B \sum_{k=0}^{n-1} \gamma^k \sum_{i=n-k+1}^{\infty} r_1^i \left[\prod_{j=1}^i \theta^{s'_{k+j}} - \prod_{j=1}^i \theta^{s_{k+j}} \right]. \quad (3.27)$$

Similarly, since $s'_{k+j} = s_{k+j}$ for $j \leq n - k$, it is possible to factor out a common term (the product over $1 \leq j \leq n - k$) from the difference in Eq.(3.27). Moreover, the product over $n - k + 1 \leq j \leq i$ is easily solved by noting that $s'_{k+j} = 1$ and $s_{k+j} = 0$:

$$\varepsilon_x(S_n) = \frac{\gamma^n}{1 - \gamma} + B \sum_{k=0}^{n-1} \gamma^k \sum_{i=n-k+1}^{\infty} r_1^i [\theta^{i-n+k} - 1] \prod_{j=1}^{n-k} \theta^{s_{k+j}}. \quad (3.28)$$

By setting $l = n - k$, $h = i - l$ and $m = l - j$, we obtain the expression

$$\varepsilon_x(S_n) = \frac{\gamma^n}{1 - \gamma} + B \gamma^n \sum_{l=1}^n \left(\frac{r_1}{\gamma} \right)^l \prod_{m=0}^{l-1} \theta^{s_{n-m}} \sum_{h=1}^{\infty} (r_2^h - r_1^h), \quad (3.29)$$

where the sum over h can easily be computed yielding, finally,

$$\varepsilon_x(S_n) = \gamma^n \left[\frac{1}{1 - \gamma} + \sum_{l=1}^n \left(\frac{r_1}{\gamma} \right)^l \prod_{m=0}^{l-1} \theta^{s_{n-m}} \right]. \quad (3.30)$$

The form of the y -contribution $\varepsilon_y(S_n)$ to the segment-length (3.25) is exactly the same as that of $\varepsilon_x(S_n)$, apart from the sign of the index j . This requires a different method for the evaluation of the sums. Since the largest value assumed by the index $k - j + 1$ in Eq.(3.23) is k , the two sequences (primed and unprimed) coincide for $k \leq n$ and give contributions which cancel each other. Therefore, with the further substitution $l = n - k$, we have

$$\varepsilon_y(S_n) = C \gamma^n \sum_{l=1}^{\infty} \gamma^l \sum_{i=1}^{\infty} p_1^i \left[\prod_{j=1}^i \pi^{s'_{l-j+n+1}} - \prod_{j=1}^i \pi^{s_{l-j+n+1}} \right]. \quad (3.31)$$

According to the value of the index $h = l - j + n + 1$, the following three situations occur:

1. for $h \geq n + 1$, $s_h = 0$ and $s'_h = 1$;
2. for $1 \leq h \leq n$, $s_h = s'_h$;
3. for $h \leq 0$, $s_h = s'_h = 0$.

Therefore, depending on the value of i , the products contain symbols belonging either to case (1) only, or to cases (1) and (2), or, finally, to all three cases. It is useful to write $\varepsilon_y(S_n) = \varepsilon_y^{(1)} + \varepsilon_y^{(2)} + \varepsilon_y^{(3)}$ and compute the three contributions separately. The first one corresponds to the interval $1 \leq i \leq l$ and is easily computed:

$$\varepsilon_y^{(1)} = \gamma^n \sum_{l=1}^{\infty} \gamma^l \left[1 - \frac{p_1^{l+2} - p_2^{l+2}}{p_1 - p_2} \right] = \gamma^{n+1} \frac{p_1 p_2}{(1 - \gamma)(1 - \gamma p_1)(1 - \gamma p_2)}. \quad (3.32)$$

In case (2), i.e. for $l+1 \leq i \leq l+n$, we set $m = i - l$ and $h = j - l$ and split the products into two parts: for $1 \leq h \leq m$, the two sequences are the same (equal to S_n) while, for $1 - l \leq h \leq 0$, we have $s'_{n+1-h} = 1$ and $s_{n+1-h} = 0$. Hence, we get

$$\varepsilon_y^{(2)} = C \gamma^n \sum_{l=1}^{\infty} \gamma^l (p_2^l - p_1^l) \sum_{m=1}^n p_1^m \prod_{h=1}^m \pi^{s_{n+1-h}}. \quad (3.33)$$

Finally, evaluating the sum over l and substituting the expression of C (3.24), we obtain

$$\varepsilon_y^{(2)} = \frac{p_1 p_2}{(1 - \gamma p_1)(1 - \gamma p_2)} \gamma^{n+1} \sum_{m=1}^n p_1^m \prod_{h=1}^m \pi^{s_{n+1-h}}. \quad (3.34)$$

The evaluation of the third contribution $\varepsilon_y^{(3)}$ requires the same procedure as the previous case with the following difference: the index i varies, now, in the range $[l+n+1, \infty)$ (i.e., $n+1 \leq m \leq \infty$) and the product over h must be interrupted at $h = n$ since, for larger h -values, the symbols s and s' are identically zero. Therefore, we have

$$\varepsilon_y^{(3)} = \frac{p_1^2}{(1 - \gamma p_1)(1 - \gamma p_2)} \gamma^{n+1} p_1^n \prod_{h=1}^n \pi^{s_{n+1-h}}. \quad (3.35)$$

This term is, however, negligible in the limit $n \rightarrow \infty$, due to the presence of the factor p_1^n . The general expression for the segment-length $\varepsilon(S_n)$ is, finally, obtained from Eqs.(3.30), (3.32) and (3.34):

$$\varepsilon(S_n) = \gamma^n \left\{ a + \sum_{l=1}^n \left(\frac{r_1}{\gamma} \right)^l \prod_{m=0}^{l-1} \theta^{s_{n-m}} + b \sum_{l=1}^n p_1^l \prod_{m=0}^{l-1} \pi^{s_{n-m}} \right\}, \quad (3.36)$$

where

$$a = \frac{1+b}{1-\gamma} \quad \text{and} \quad b = \frac{\gamma p_1 p_2}{(1 - \gamma p_1)(1 - \gamma p_2)}. \quad (3.37)$$

Notice that the segment-length $\varepsilon(S_n)$ effectively depends on the symbol sequence $S_n = \{s_1, \dots, s_n\}$ only (the time-direction has been reverted, in Eq.(3.11), in order to obtain positive indices in the final expression (3.36)). For the analysis of the first transition, which occurs at the point

$$\ln \gamma = p_1 \ln r_1 + p_2 \ln r_2 = \langle \lambda_2 \rangle, \quad (3.38)$$

for $q = 1$, we can simplify the calculations by choosing $p_1 = p_2 = 1/2$. This constitutes, actually, no restriction, since the non-uniformity of the attractor is guaranteed by the fluctuations of the second multiplier (either equal to r_1 or to r_2) and the presence of the filter affects only the corresponding sum in Eq.(3.36) (the second one converges to a constant, for $n \rightarrow \infty$).

If the filter bandwidth is large (i.e., if γ is smaller than both r_1 and r_2), the first sum in Eq.(3.36) diverges, for large n , and the constant can be neglected. It is then easy to see that the asymptotic behaviour of $\varepsilon(S_n)$ is the same as that of the unfiltered segment-lengths in the x -direction: for example, all sequences ending with an infinite string of 0's yield $\varepsilon(S_n) \sim r_1^n$ and those ending with 1's yield $\varepsilon(S_n) \sim r_2^n$. All other sequences (characterized by different percentages of 0's and 1's) present an intermediate behaviour between these two. The pointwise dimension is given by

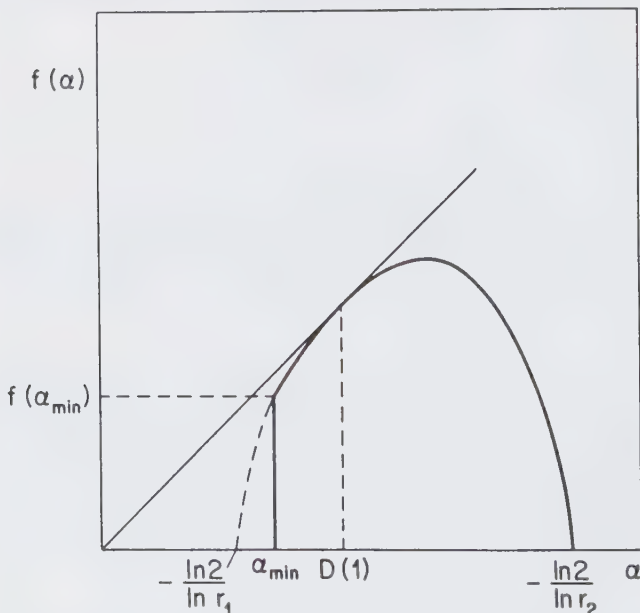


Figure 20: Schematic plot of the $f(\alpha)$ -curve (solid line) for the filtered baker map in the parameter region a of Fig. 16, with $r_1 < \gamma < \sqrt{r_1 r_2}$. The dashed line represents the same curve in absence of filter.

the ratio $\ln P(S_n) / \ln \varepsilon(S_n)$, with $P(S_n) = 2^{-n}$ (since $p_1 = p_2 = 1/2$) and belongs to the interval $[\alpha_{\min} \equiv D(+\infty) = -\ln 2 / \ln r_1, \alpha_{\max} \equiv D(-\infty) = -\ln 2 / \ln r_2]$ (assuming, e.g., $r_1 < r_2$). When γ is increased above r_1 , the asymptotic behaviour of

$\varepsilon(S_n)$ changes: in fact, depending on the sequence S_n , either it exhibits the same scaling as before (powers of r_1 and r_2), or it decreases like γ^n (as, for example, all sequences ending with 1's). Therefore, the smallest pointwise dimension $\alpha_{\min}$ becomes now $\alpha_{\min}(\gamma) = -\ln 2 / \ln \gamma$ and increases with increasing γ . All points (i.e., sequences) which, in absence of filtering, were characterized by a pointwise dimension between $-\ln 2 / \ln r_1$ and the value $-\ln 2 / \ln \gamma$, exhibit now a dimension equal to the new $\alpha_{\min}(\gamma)$. This phenomenon is illustrated in Fig. 20: the curve $f(\alpha)$ vs. α (solid line) coincides with the unperturbed one (dashed line), for α larger than $\alpha_{\min}(\gamma)$, while displaying a cut-off at $\alpha = \alpha_{\min}(\gamma)$. The Hausdorff dimension of the set U of all points which have a pointwise dimension $\alpha = \alpha_{\min}$ is $f(\alpha_{\min})$ (see Section 2.2). Indeed, U is the union of all sets whose points, without filter, had a pointwise dimension $\alpha \leq \alpha_{\min}$. Each of those sets has a Hausdorff dimension equal to $f(\alpha)$ and the dimension of their union U is the largest of all $f(\alpha)$ -values, for $\alpha \leq \alpha_{\min}$ (see Section 3.1). The function $f(\alpha)$ jumps from the

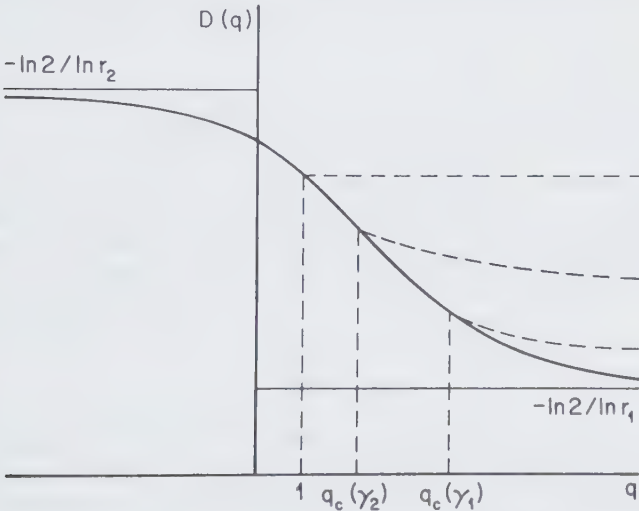


Figure 21: Dashed lines: dimension function $D(q)$ vs. q , for some values of the parameter $\gamma \in (0, \sqrt{r_1 r_2})$. Solid line: $D(q)$ in absence of filter.

value $f(\alpha_{\min})$ (at $\alpha = \alpha_{\min}$) to $-\infty$, for $\alpha < \alpha_{\min}$. Hence, from Eq.(2.9), we obtain

$$D(q) \sim \frac{q\alpha_{\min} - f(\alpha_{\min})}{q - 1}, \quad (3.39)$$

for $q > q_c$, where the critical q is $q_c = \lim_{\alpha \searrow \alpha_{\min}} f'(\alpha)$. This behaviour is similar to that found for the logistic map, where the transition was at $q = q_c = 2$, with $\alpha_{\min} =$

$1/2$ and $f(\alpha_{\min}) = 0$ (see Fig. 14). Of course, both the critical “temperature” q_c and the value $\alpha_{\min}$ are functions of the external parameter γ (an analytical evaluation of them is, however, rather involved, except for special γ -values). The dimension function $D(q)$ is unchanged, for $q \leq q_c(\gamma)$.

A particularly interesting case is found for the value of γ which yields a transition at $q_c(\gamma) = 1$, corresponding to $\gamma = \sqrt{r_1 r_2}$ (see Eq.(3.38)), for $p_1 = p_2 = 1/2$. At that point, $\alpha_{\min} = f(\alpha_{\min}) = D(1)$ and $D(q) = D(1)$ for any $q \geq q_c$ (see Fig. 20 and Eq.(3.39)). The behaviour of the dimension function $D(q)$, for various γ -values, is schematically displayed in Fig. 21.

According to the discussion of Part II, the hamiltonian can be defined as the logarithm of the generic segment-length (3.36). Alternatively, in order to obtain a more asymptotic estimate, we could concentrate on a subset of the attractor of size $\varepsilon(S_n)$ and consider its internal structure (which exhibits the same features of the whole set). In this case, we should rescale the lengths of all segments of order $n+1$ to the size $\varepsilon(S_n)$ of that subset, by taking the ratio $\hat{\sigma}(S_{n+1}) = \varepsilon(S_{n+1})/\varepsilon(S_n)$ (see Eq.(2.18)). Although this latter method presents a faster numerical convergence, we follow the former one, which yields a simpler expression for the hamiltonian $H(S_n)$, defined as

$$H(S_n) \equiv \ln \varepsilon(S_n) = n \ln \gamma + \ln \left[c + \sum_{l=1}^n \left(\frac{\sqrt{r_1 r_2}}{\gamma} \right)^l \prod_{m=0}^{l-1} \theta^{\sigma_{n-m}/2} \right], \quad (3.40)$$

where the constant c is, from Eqs.(3.36) and (3.37), $c = a + b$ ($p_1 = p_2$) and the symmetrized form has been used for the spins: $\sigma_i = 2s_i - 1 = \pm 1$. By noticing that $\theta^{\sigma/2}$ can be rewritten as $(\frac{1+\sigma}{2})\theta^{1/2} + (\frac{1-\sigma}{2})\theta^{-1/2}$, it is possible to transform the sum in Eq.(3.40) in such a way that the spin-values σ no longer appear in the exponent. The final form of the hamiltonian is the following:

$$H = h_0 + \sum_{i=1}^n J_i^{(1)} \sigma_i + \frac{1}{2} \sum_{i \neq j=1}^n J_{ij}^{(2)} \sigma_i \sigma_j + \frac{1}{6} \sum_{i \neq j \neq k=1}^n J_{ijk}^{(3)} \sigma_i \sigma_j \sigma_k + \dots + J^{(n)} \sigma_1 \dots \sigma_n, \quad (3.41)$$

where h_0 is a constant and the interaction coefficients J explicitly depend on the site i (i.e., they are not translationally invariant). This situation is common in dynamical systems and has also been observed in Refs.[61,85]. However, it is possible to find translationally invariant interactions which lead to the same total hamiltonian by summing all coefficients of the same kind (1-spin interactions, 2-spin interactions, etc.) [61,85]. The calculation is rather tedious, although straightforward, and the expressions of the coefficients long, so that they will not be given here. The magnitude of the J 's does not depend on the distance (e.g., $J_{1,2}^{(2)}$ is of the same order of $J_{1,n}^{(2)}$). The sign of the two-spin terms $J_{ij}^{(2)}$ is negative for $\gamma < \sqrt{r_1 r_2}$ and positive for $\gamma > \sqrt{r_1 r_2}$, indicating a passage from ferromagnetic to antiferromagnetic coupling. At the special value $\gamma = \sqrt{r_1 r_2}$, all even-order interactions vanish, for $n \rightarrow \infty$. Of course, by using the set of definitions (2.4), it is possible to pass from antiferromagnetic to ferromagnetic coupling. Notice that the interactions are between all orders of nearest-neighbours ($1^{st}, 2^{nd}, \dots, n^{th}$), and involve up

to n spins: these long-range interactions are responsible for the occurrence of the phase-transition illustrated in Figs. 20 and 21. Furthermore, a critical slowing-down effect exists, as it can be seen from Eq.(3.40). In fact, in order to compute the information dimension $D(1) = -n \ln 2 / \langle \ln \varepsilon(S_n) \rangle$, for example, we have to evaluate the average value of the hamiltonian (3.40), over all symbol sequences (we consider, for simplicity, the case $\gamma = \sqrt{r_1 r_2}$):

$$\langle H(S_n) \rangle = n \ln \gamma + \left\langle \ln \left[c + \sum_{l=1}^n e^{k \sum_{m=0}^{l-1} \sigma_{n-m}} \right] \right\rangle, \quad (3.42)$$

where $k = \frac{1}{2} \ln \theta$. Notice that σ_i , for $i = 1, \dots, n$, represents a discrete-time random process ($\sigma = \pm 1$, with equal probability) and $W_l \equiv \sum_{m=0}^{l-1} \sigma_{n-m}$ is equivalent to a brownian motion. The sum in Eq.(3.42) is bounded between $\exp[kW_{\max}(n)]$ and $n \exp[kW_{\max}(n)]$, where $W_{\max}(n)$ is the largest value reached by W_l in n steps (i.e., for $1 \leq l \leq n$), so that we can neglect the constant c . Then, we obtain

$$\langle H(S_n) \rangle \sim n \ln \gamma + \langle kW_{\max}(n) \rangle,$$

apart from correction terms of the order of $\ln n$. Recalling that $\langle kW_{\max}(n) \rangle \sim \sqrt{n}$ for large n [86], we see that the convergence of $\langle H \rangle$ to the asymptotic behaviour $n \ln \gamma$ is slowed down by the presence of the “diffusive” term proportional to $\sqrt{n}$.

The long-memory effects induced by the filtering can be seen from the expression of the hamiltonian, which clearly depends on the whole sequence S_n : in particular, the dependence on the last symbols is stronger (e.g., σ_n appears in all terms of the sum in Eq.(3.40), while σ_1 enters only once, for $l = n$). Such long-time correlations are related to the occurrence of tangencies between the eigendirections in the contracting subspace $x-z$. In fact, the linearized matrix of the n -th iteration of the map, given by products of single-step matrices of the form

$$\begin{pmatrix} r_\nu & 0 & 0 \\ 0 & p_\nu^{-1} & 0 \\ 1 & 1 & \gamma \end{pmatrix}, \quad \text{with } \nu = 1, 2, \quad (3.43)$$

has eigenvectors whose components in the $x-z$ plane fluctuate, depending on the value of the second multiplier $\lambda_2(n) = r_1^i r_2^{n-i}$ (i is the number of times the symbol 0 occurred in n iterations). If γ is larger than r_1 , the two eigenvectors can point in nearly the same direction, even for large n , for certain symbol sequences, thus yielding correlations between the structure of the attractor along the x - and the z -direction. This behaviour is, clearly, possible only for three- or higher-dimensional (hyperbolic) maps. In the non-hyperbolic case, two dimensions are sufficient (e.g., Hénon’s map), due to the homoclinic tangencies.

The analysis performed above in terms of segment-lengths could also be done by using relation (1.52), which connects dimensions to Lyapunov exponents and holds exactly in this case, since the system is hyperbolic. Then we would observe that the distribution of the second Lyapunov exponent $h(\lambda_2)$, which is smooth and

bell-shaped in absence of filtering, presents a cut-off at the value $\lambda_2 = \ln \gamma$ and is not altered for $\lambda_2 > \ln \gamma$, in analogy with the behaviour of $f(\alpha)$ (see Fig. 20).

The transition can also occur for $q_c < 1$: indeed, the $f(\alpha)$ -curve displays a cut-off at larger and larger α -values, until it becomes a delta-function (terminating at $f(\alpha) = \alpha$), for $\gamma = r_2$. At that point, the dimension function $D(q)$ becomes a constant, equal to $-\ln 2 / \ln \gamma$.

At the second transition point T_2 (see Fig. 16), the information dimension $D(1)$ increases beyond 2 and the projection of the sections of Figs. 17-19 onto the z -axis are no more one-to-one. In the non-uniform case, this phenomenon does not occur simultaneously at all length-scales and for all the segments, upon increase of γ , as in the uniform case. Moreover, no discontinuity in the dimension function $D(q)$ appears, showing that this transition (related to the fluctuations of the first Lyapunov exponent) has a continuous character.

Finally, it is worth mentioning that phase-transition-like behaviour could also be obtained for the metric entropy $K(q)$ in systems with more than one Lyapunov exponent (one of which should be fluctuating). In all these cases, a prominent role is played by slowly-decaying correlations, caused by the long-time coincidence of eigendirections in the linearized matrix [82].

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